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Rationality Problems in Algebraic Geometry

Levico Terme, Italy 2015

Rita Pardini · Gian Pietro Pirola *Editors*

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Introduction

Historically, rationality problems have been of great importance in Algebraic Geometry and have motivated fundamental developments, as the theory of abelian integrals, Riemann surfaces, and the Abel–Jacobi map; Castelnuovo's solution of the Lüroth problem in dimension 2 showed the power of the geometric methods and opened the way to the Enriques classification of surfaces; at the end of the 1960s the counterexamples to the Lüroth problem in dimension 3, by Clemens–Griffiths, Iskovskih–Manin, and Artin–Mumford, established the importance of Griffiths theory of periods, of the birational geometry of the Cremona, and of the Brauer group; very recently Voisin introduced a new method to study stable rationality via an analysis of the class of the diagonal in the Chow group of the self-product of a variety with itself.

Finally, as it is well illustrated by the Harris–Mumford work on the Severi conjecture, the rationality problems had also a prominent position in the development of moduli theory.

In higher dimension, rationality problems are widely open. While it is not easy to make any forecast about definitive progress, the last years have witnessed a lot of ingenious new ideas being inserted into the classical picture, and this gives high hopes that some real advance is boiling in the pot. Many efforts have been made in the study of special Hodge structures, on the Cremona Group, and new methods (e.g., derived category) have been introduced, new conjectures have been formulated.

The CIME–CIRM course *Rationality Problems in Algebraic Geometry* was organized with the aim to underline the emerging trends and stimulate young researchers to get involved in this fascinating area. It took place in Levico from June 22 to June 27, 2016, and was attended by about 65 people. It consisted of three sets of lectures, delivered by Arnaud Beauville (*The Lüroth problem*), Alexander Kuznetsov (*Derived category view on rationality problems*), and Alessandro Verra (*Classical moduli spaces and rationality*), complemented by three seminars on related topics by Ciro Ciliberto, Howard Nuer, and Paolo Stellari. The original plan of the course included lectures by Brendan Hassett (*Cubic fourfolds, K*3 *surfaces,* *and rationality questions*) who was unable to come to the course but contributed his notes. This volume contains the written notes by Beauville, Kuznetsov, Verra, and Hasset and a write up of Nuer's talk.

We wish to thank first of all CIME and CIRM, whose financial support made the course possible. Then we would like to thank the lecturers and all the participants for the very pleasant atmosphere created during the course: although most (including the organizers) were nonexpert in the field, all took part in the activities with great enthusiasm and benefit. Finally, we would like to acknowledge the help of Mr. Augusto Micheletti who took excellent care of the practical aspects of the organization.

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The Lüroth Problem

Arnaud Beauville

Abstract The Lüroth problem asks whether every unirational variety is rational. After a historical survey, we describe the methods developed in the 1970s to get a negative answer, and give some easy examples. Then we discuss a new method introduced last year by C. Voisin.

1 Some History

1.1 Curves and Surfaces

In 1876 appears a three pages note by J. Lüroth [L], where he proves that if a complex algebraic curve C can be parametrized by rational functions, one can find another parametrization which is generically one-to-one. In geometric language, if we have a dominant rational map $f : \mathbb{P}^1 \dashrightarrow C$, then C is a rational curve.

By now this is a standard exercise: we can assume that C is smooth projective, then f is a morphism, which induces an injective homomorphism $f^* : H^0(C, \Omega^1_C) \to H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$. Thus C has no nontrivial holomorphic 1-form, hence has genus 0, and this implies $C \cong \mathbb{P}^1$.

Actually Lüroth does not mention at all Riemann surfaces, but uses instead an ingenious and somewhat sophisticated algebraic argument. I must say that I find somewhat surprising that he did not consider applying Riemann's theory, which had appeared 20 years before.

Anyhow, clearly Lüroth's paper had an important impact. When Castelnuovo and Enriques develop the theory of algebraic surfaces, in the last decade of the nineteenth century, one of the first questions they attack is whether the analogous statement holds for surfaces. Suppose we have a smooth projective surface S (over $\mathbb{C}$) and a dominant rational map $f : \mathbb{P}^2 \dashrightarrow S$. As in the curve case, this implies $H^0(S, \Omega^1_S) = H^0(S, K_S) = 0$ (note that f is well-defined outside a finite subset). At first Castelnuovo hoped that this vanishing would be sufficient to characterize

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rational surfaces, but Enriques suggested a counter-example, now known as the Enriques surface. Then Castelnuovo found the right condition, namely $H^0(S, \Omega^1_S) = H^0(S, K^2_S) = 0$; this is satisfied by our surface S, and Castelnuovo proves that it implies that S is rational. After more than one century, even if the proof has been somewhat simplified, this is still a highly nontrivial result.

1.2 Attempts in Dimension 3

At this point it becomes very natural to ask what happens in higher dimension. Let us first recall the basic definitions (see Sect. 3 for a more elaborate discussion): a complex variety X of dimension n is *unirational* if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$; it is *rational* if there is a birational such map. The *Lüroth problem* asks whether every unirational variety is rational.

In 1912, Enriques proposed a counter-example in dimension 3 [E], namely a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$—we will use the notation $V_{2,3}$ for such a complete intersection. Actually what Enriques does in this two pages paper is to prove the unirationality of $V_{2,3}$, in a clever (and correct) way; for the non-rationality he refers to a 1908 paper by Fano [F1].

In the course of his thorough study of what we now call Fano manifolds, Fano made various attempts to prove that some of them are not rational [F2, F4]. Unfortunately the birational geometry of threefolds is considerably more complicated than that of surfaces; while the intuitive methods of the Italian geometers were sufficient to handle surfaces, they could not treat adequately higher-dimensional manifolds. None of Fano's attempted proofs is acceptable by modern standards.

A detailed criticism of these attempts can be found in the book [R]. It is amusing that after concluding that none of them can be considered as correct, Roth goes on and proposes a new counter-example, which is not simply connected and therefore not rational (the fundamental group is a birational invariant). Alas, a few years later Serre (motivated in part by Roth's claim) proved that a unirational variety is simply connected [S].

1.3 The Modern Era

Finally, in 1971–1972, three different (indisputable) counter-examples appeared. We will discuss at length these results in the rest of the paper; let us indicate briefly here the authors, their examples and the methods they use to prove non-rationality:

Authors	Example	Method
Clemens-Griffiths	$V_3 \subset \mathbb{P}^4$	JV
Iskovskikh-Manin	Some $V_4 \subset \mathbb{P}^4$	Bir(V)
Artin-Mumford	Specific	Tors $H^3(V, \mathbb{Z})$

More precisely:

- Clemens-Griffiths [C-G] proved the longstanding conjecture that a smooth cubic threefold $V_3 \subset \mathbb{P}^4$ is not rational—it had long been known that it is unirational. They showed that the *intermediate Jacobian* of V_3 is not a Jacobian (*Clemens-Griffiths criterion*, see Theorem 1 below).
- Iskovskikh-Manin [I-M] proved that any smooth quartic threefold $V_4 \subset \mathbb{P}^4$ is not rational. Some unirational quartic threefolds had been constructed by B. Segre [Sg2], so these provide counter-examples to the Lüroth problem. They showed that the group of birational automorphisms of V_4 is finite, while the corresponding group for $\mathbb{P}^3$ is huge.
- Artin-Mumford [A-M] proved that a particular double covering X of $\mathbb{P}^3$, branched along a quartic surface in $\mathbb{P}^3$ with ten nodes, is unirational but not rational. They showed that the torsion subgroup of $H^3(X, \mathbb{Z})$ is nontrivial, and is a birational invariant.

These three papers have been extremely influential. Though they appeared around the same time, they use very different ideas; in fact, as we will see, the methods tend to apply to different types of varieties. They have been developed and extended, and applied to a number of interesting examples. Each of them has its advantages and its drawbacks; very roughly:

- The intermediate Jacobian method is quite efficient, but applies only in dimension 3;
- The computation of birational automorphisms leads to the important notion of *birational rigidity*. However it is not easy to work out; so far it applies essentially to Fano varieties of index 1 (see Sect. 2.3), which are not known to be unirational in dimension > 3.
- Torsion in H^3 gives an obstruction to a property weaker than rationality, called *stable rationality* (Sect. 5). Unfortunately it applies only to very particular varieties, and not to the standard examples of unirational varieties, like hypersurfaces or complete intersections. However we will discuss in Sect. 7 a new idea of C. Voisin which extends considerably the range of that method.

They are still essentially the basic methods to prove non-rationality results. A notable exception is the method of Kollár using reduction modulo p; however it applies only to rather specific examples, which are not known to be unirational. We will describe briefly his results in Sect. 4.2.

A final remark: at the time they were discovered the three methods used the difficult *resolution of indeterminacies* due to Hironaka. This is a good reason why the Italian algebraic geometers could not succeed! It was later realized that the birational invariance of $\operatorname{Tors} H^3(V, \mathbb{Z})$ can be proved without appealing to the resolution of singularities, see Sect. 6.4—but this still requires some highly nontrivial algebraic apparatus.

2 The Candidates

In this section we will introduce various classes of varieties which are natural candidates to be counter-examples to the Lüroth problem.

2.1 Rationality and Unirationality

Let us first recall the basic definitions which appear in the Lüroth problem. We work over the complex numbers. A *variety* is an integral scheme of finite type over $\mathbb{C}$.

Definition 1

1) A variety V is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow V$.
2) V is *rational* if there exists a birational map $\mathbb{P}^n \overset{\sim}{\dashrightarrow} V$.

In the definition of unirationality we can take $n = \dim V$: indeed, if we have a dominant rational map $\mathbb{P}^N \dashrightarrow V$, its restriction to a general linear subspace of dimension $\dim(V)$ is still dominant.

We may rephrase these definitions in terms of the function field $\mathbb{C}(V)$ of V: V is unirational if $\mathbb{C}(V)$ is contained in a purely transcendental extension of $\mathbb{C}$; V is rational if $\mathbb{C}(V)$ is a purely transcendental extension of $\mathbb{C}$. Thus the Lüroth problem asks whether every extension of $\mathbb{C}$ contained in $\mathbb{C}(t_1, \ldots, t_n)$ is purely transcendental.

2.2 Rational Connectedness

Though the notion of unirationality is quite natural, it is very difficult to handle. The crucial problem is that so far there is no known method to prove non-unirationality, like the ones we mentioned in Sect. 1.3 for non-rationality.

There is a weaker notion which behaves much better than unirationality, and which covers all varieties we will be interested in:

Definition 2 A smooth projective variety V is *rationally connected* (RC for short) if any two points of V can be joined by a rational curve.

It is enough to ask that two general points of V can be joined by a rational curve, or even by a chain of rational curves. In particular, rational connectedness is a birational property.

In contrast to unirationality, rational connectedness has extremely good properties (see for instance [Ar] for proofs and references):

a) It is an open and closed property; that is, given a smooth projective morphism $f : V \to B$ with B connected, if some fiber of f is RC, *all* the fibers are RC.
b) Let $f : V \dashrightarrow B$ be a rational dominant map. If B and the general fibers of f are RC, V is RC.
c) If V is RC, all contravariant tensor fields vanish; that is, $H^0(V, (\Omega^1_V)^{\otimes n}) = 0$ for all n. It is conjectured that the converse holds; this is proved in dimension ≤ 3.

Neither *a*) nor *b*) [nor, a fortiori, *c*)] are expected to hold when we replace rational connectedness by unirationality or rationality. For *a*), it is expected that the general quartic threefold is *not* unirational (see [R, V.9]), though some particular V_4 are; so unirationality should not be stable under deformation. Similarly it is expected that the general cubic fourfold is not rational, though some of them are known to be rational.

Projecting a cubic threefold V_3 from a line contained in V_3 gives a rational dominant map to $\mathbb{P}^2$ whose generic fiber is a rational curve, so *b*) does not hold for rationality. The same property holds more generally for a general hypersurface of degree d in $\mathbb{P}^4$ with a $(d-2)$-uple line; it is expected that it is not even unirational for $d \geq 5$ [R, IV.6].

2.3 Fano Manifolds

A more restricted class than RC varieties is that of *Fano manifolds*—which were extensively studied by Fano in dimension 3. A smooth projective variety V is *Fano* if the anticanonical bundle K_V^{-1} is ample. This implies that V is RC; but contrary to the notions considered so far, this is not a property of the birational class of V.

A Fano variety V is called *prime* if $\mathrm{Pic}(V) = \mathbb{Z}$ (the classical terminology is “of the first species”). In that case we have $K_V = L^{-r}$, where L is the positive generator of $\mathrm{Pic}(V)$. The integer r is called the *index* of V. Prime Fano varieties are somehow minimal among RC varieties: they do not admit a Mori type contraction or morphisms to smaller-dimensional varieties.

In the following table we list what is known about rationality issues for prime Fano threefolds, using their classification by Iskovskikh [I1]: for each of them, whether it is unirational or rational, and, if it is not rational, the method of proof and the corresponding reference. The only Fano threefolds of index ≥ 3 are $\mathbb{P}^3$ and the smooth quadric $V_2 \subset \mathbb{P}^4$, so we start with index 2, then 1:

Variety	Unirational	Rational	Method	Reference
$V_6 \subset \mathbb{P}(1,1,1,2,3)$	?	No	$\mathrm{Bir}(V)$	[Gr]
Quartic double $\mathbb{P}^3$	Yes	No	JV	[V1]
$V_3 \subset \mathbb{P}^4$	"	No	JV	[C-G]
$V_{2,2} \subset \mathbb{P}^5$, $X_5 \subset \mathbb{P}^6$	"	Yes		
Sextic double $\mathbb{P}^3$	?	No	$\mathrm{Bir}(V)$	[I-M]
$V_4 \subset \mathbb{P}^4$	Some	No	$\mathrm{Bir}(V)$	[I-M]
$V_{2,3} \subset \mathbb{P}^5$	Yes	No (generic)	JV, $\mathrm{Bir}(V)$	[B1, P]
$V_{2,2,2} \subset \mathbb{P}^6$	"	No	JV	[B1]
$X_{10} \subset \mathbb{P}^7$	"	No (generic)	JV	[B1]
$X_{12}, X_{16}, X_{18}, X_{22}$	"	Yes		
$X_{14} \subset \mathbb{P}^9$	"	No	JV	[C-G] + [F3][1]

A few words about notation: as before $V_{d_1,\ldots,d_p}$ denotes a smooth complete intersection of multidegree $(d_1, \ldots, d_p)$ in $\mathbb{P}^{p+3}$, or, for the first row, in the weighted projective space $\mathbb{P}(1,1,1,2,3)$. A quartic (resp. sextic) double $\mathbb{P}^3$ is a double cover of $\mathbb{P}^3$ branched along a smooth quartic (resp. sextic) surface. The notation $X_d \subset \mathbb{P}^m$ means a smooth threefold of degree d in $\mathbb{P}^m$. The mention "(generic)" means that non-rationality is known only for those varieties belonging to a certain Zariski open subset of the moduli space.

2.4 Linear Quotients

An important source of unirational varieties is provided by the quotients V/G, where G is an algebraic group (possibly finite) acting linearly on the vector space V. These varieties, and the question whether they are rational or not, appear naturally in various situations. The case G finite is known as the Noether problem (over $\mathbb{C}$); we will see below (Sect. 6.4) that a counter-example has been given by Saltman [Sa], using an elaboration of the Artin-Mumford method. The case where G is a connected linear group appears in a number of moduli problems, but there is still no example where the quotient V/G is known to be non-rational—in fact the general expectation is that all these quotients should be rational, but this seems out of reach at the moment.

A typical case is the moduli space $\mathscr{H}_{d,n}$ of hypersurfaces of degree $d \geq 3$ in $\mathbb{P}^n$, which is birational to $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))/\mathrm{GL}_{n+1}$—more precisely, it is the quotient of the open subset of forms defining a smooth hypersurface. For $n = 2$ the rationality is now known except for a few small values of d, see for instance [BBK] for an up-to-date summary; for $n \geq 3$ there are only a few cases where $\mathscr{H}_{d,n}$ is known to

[1] Fano proved in [F3] that the variety X_{14} is birational to a smooth cubic threefold.

be rational. We refer to [D] for a survey of results and problems, and to [C-S] for a more recent text.

3 The Intermediate Jacobian

In this section we discuss our first non-rationality criterion, using the intermediate Jacobian. Then we will give an easy example of a cubic threefold which satisfies this criterion, hence gives a counter-example to the Lüroth problem.

3.1 The Clemens-Griffiths Criterion

In order to define the intermediate Jacobian, let us first recall the Hodge-theoretic construction of the Jacobian of a (smooth, projective) curve C. We start from the Hodge decomposition

$$H^1(C,\mathbb{Z}) \subset H^1(C,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

with $H^{0,1} = \overline{H^{1,0}}$. The latter condition implies that the projection $H^1(C,\mathbb{R}) \to H^{0,1}$ is a ($\mathbb{R}$-linear) isomorphism, hence that the image Γ of $H^1(C,\mathbb{Z})$ in $H^{0,1}$ is a *lattice* (that is, any basis of Γ is a basis of $H^{0,1}$ over $\mathbb{R}$). The quotient $JC := H^{0,1}/\Gamma$ is a complex torus. But we have more structure. For $\alpha,\beta \in H^{0,1}$, put $H(\alpha,\beta) = 2i\int_C \bar{\alpha}\wedge\beta$. Then H is a positive hermitian form on $H^{0,1}$, and the restriction of $\operatorname{Im}(H)$ to $\Gamma \cong H^1(C,\mathbb{Z})$ coincides with the cup-product

$$H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z}) \to H^2(C,\mathbb{Z}) = \mathbb{Z}\ ;$$

thus it induces on Γ a skew-symmetric, integer-valued form, which is moreover *unimodular*. In other words, H is a *principal polarization* on JC (see [B-L], or [B5] for an elementary treatment). This is equivalent to the data of an ample divisor $\Theta \subset JC$ (defined up to translation) satisfying $\dim H^0(JC,\mathcal{O}_{JC}(\Theta)) = 1$. Thus (JC,Θ) is a *principally polarized abelian variety* (p.p.a.v. for short), called the *Jacobian* of C.

One can mimic this definition for higher dimensional varieties, starting from the odd degree cohomology; this defines the general notion of intermediate Jacobian. In general it is only a complex torus, not an abelian variety. But the situation is much nicer in the case of interest for us, namely rationally connected threefolds. For such a threefold V we have $H^{3,0}(V) = H^0(V,K_V) = 0$, hence the Hodge decomposition for H^3 becomes:

$$H^3(V,\mathbb{Z})_{\mathrm{tf}} \subset H^3(V,\mathbb{C}) = H^{2,1} \oplus H^{1,2}$$

with $H^{1,2} = \overline{H^{2,1}}$ ($H^3(V,\mathbb{Z})_{\text{tf}}$ denotes the quotient of $H^3(V,\mathbb{Z})$ by its torsion subgroup). As above $H^{1,2}/H^3(V,\mathbb{Z})_{\text{tf}}$ is a complex torus, with a principal polarization defined by the hermitian form $(\alpha,\beta) \mapsto -2i\int_V \bar{\alpha}\wedge\beta$: this is the *intermediate Jacobian JV* of V.

We will use several times the following well-known and easy lemma, see for instance [V2, Theorem 7.31]:

Lemma 1 *Let X be a complex manifold, $Y \subset X$ a closed submanifold of codimension c, $\hat{X}$ the variety obtained by blowing up X along Y. There are natural isomorphisms*

$$H^p(\hat{X},\mathbb{Z}) \xrightarrow{\sim} H^p(X,\mathbb{Z}) \oplus \sum_{k=1}^{c-1} H^{p-2k}(Y,\mathbb{Z}) \ .$$

Theorem 1 (Clemens-Griffiths Criterion) *Let V be a smooth rational projective threefold. The intermediate Jacobian JV is isomorphic (as p.p.a.v.) to the Jacobian of a curve or to a product of Jacobians.*

Sketch of Proof Let $\varphi : \mathbb{P}^3 \overset{\sim}{\dashrightarrow} V$ be a birational map. Hironaka's resolution of indeterminacies provides us with a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "b"'] \arrow[dr, "f"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\varphi"] & & V \end{tikzcd}$$

where $b : P \to \mathbb{P}^3$ is a composition of blowing up, either of points or of smooth curves, and f is a birational *morphism*.

We claim that JP is a product of Jacobians of curves. Indeed by Lemma 1, blowing up a point in a threefold V does not change $H^3(V,\mathbb{Z})$, hence does not change JV either. If we blow up a smooth curve $C \subset V$ to get a variety $\hat{V}$, Lemma 1 gives a canonical isomorphism $H^3(\hat{V},\mathbb{Z}) \cong H^3(V,\mathbb{Z}) \oplus H^1(C,\mathbb{Z})$, compatible in an appropriate sense with the Hodge decomposition and the cup-products; this implies $J\hat{V} \cong JV \times JC$ as p.p.a.v. Thus going back to our diagram, we see that JP is isomorphic to $JC_1 \times \ldots \times JC_p$, where $C_1,\ldots,C_p$ are the (smooth) curves which we have blown up in the process.

How do we go back to JV? Now we have a birational *morphism* $f : P \to V$, so we have homomorphisms $f^* : H^3(V,\mathbb{Z}) \to H^3(P,\mathbb{Z})$ and $f_* : H^3(P,\mathbb{Z}) \to H^3(V,\mathbb{Z})$ with $f_*f^* = 1$, again compatible with the Hodge decomposition and the cup-products in an appropriate sense. Thus $H^3(V,\mathbb{Z})$, with its polarized Hodge structure, is a direct factor of $H^3(P,\mathbb{Z})$; this implies that JV is a direct factor of $JP \cong JC_1 \times \ldots \times JC_p$, in other words there exists a p.p.a.v. A such that $JV \times A \cong JC_1 \times \ldots \times JC_p$.

How can we conclude? In most categories the decomposition of an object as a product is not unique (think of vector spaces!). However here a miracle occurs. Let us say that a p.p.a.v. is *indecomposable* if it is not isomorphic to a product of nontrivial p.p.a.v.

Lemma 2

1) *A p.p.a.v.* (A, Θ) *is indecomposable if and only if the divisor* Θ *is irreducible.*
2) *Any p.p.a.v. admits a* unique *decomposition as a product of indecomposable p.p.a.v.*

Sketch of Proof We start by recalling some classical properties of abelian varieties, for which we refer to [M]. Let D be a divisor on an abelian variety A; for $a \in A$ we denote by D_a the translated divisor $D + a$. The map $\varphi_D : a \mapsto \mathscr{O}_A(D_a - D)$ is a homomorphism from A into its dual variety $\hat{A}$, which parametrizes topologically trivial line bundles on A. If D defines a principal polarization, this map is an isomorphism.

Now suppose our p.p.a.v. (A, Θ) is a product $(A_1, \Theta_1) \times \ldots \times (A_p, \Theta_p)$. Then $\Theta = \Theta^{(1)} + \ldots + \Theta^{(p)}$, with $\Theta^{(i)} := A_1 \times \ldots \Theta_i \times \ldots \times A_p$; we recover the summand $A_i \subset A$ as $\varphi_\Theta^{-1}(\varphi_{\Theta^{(i)}}(A))$. Conversely, let (A, Θ) be a p.p.a.v., and let $\Theta^{(1)}, \ldots, \Theta^{(p)}$ be the irreducible components of Θ (each of them occurs with multiplicity one, since otherwise one would have $h^0(A; \mathscr{O}_A(\Theta)) > 1$). Putting $A_i := \varphi_\Theta^{-1}(\varphi_{\Theta^{(i)}}(A))$ and $\Theta_i := \Theta^{(i)}_{|A_i}$, it is not difficult to check that (A, Θ) is the product of the (A_i, Θ_i) —see [C-G], Lemma 3.20 for the details. □

Once we have this, we conclude as follows. The Theta divisor of a Jacobian JC is the image of the Abel-Jacobi map $C^{(g-1)} \to JC$, and therefore is irreducible. From the isomorphism $JV \times A \cong JC_1 \times \ldots \times JC_p$ and the Lemma we conclude that JV is isomorphic to $JC_{i_1} \times \ldots \times JC_{i_r}$ for some subset $\{i_1, \ldots, i_r\}$ of $[1, p]$. □

Remark One might think that products of Jacobians are more general than Jacobians, but it goes the other way around: in the moduli space $\mathscr{A}_g$ of g-dimensional p.p.a.v., the boundary $\bar{\mathscr{J}}_g \smallsetminus \mathscr{J}_g$ of the Jacobian locus is precisely the locus of products of lower-dimensional Jacobians.

3.2 *The Schottky Problem*

Thus to show that a threefold V is not rational, it suffices to prove that its intermediate Jacobian is not the Jacobian of a curve, or a product of Jacobians. Here we come across the classical Schottky problem: the characterization of Jacobians among all p.p.a.v. (the usual formulation of the Schottky problem asks for equations of the Jacobian locus inside the moduli space of p.p.a.v.; here we are more interested in special geometric properties of Jacobians). One frequently used approach is through the singularities of the Theta divisor: the dimension of $\mathrm{Sing}(\Theta)$ is $\geq g - 4$ for a Jacobian (JC, Θ) of dimension g, and $g - 2$ for a product. However controlling $\mathrm{Sing}(\Theta)$ for an intermediate Jacobian is quite difficult, and requires a lot of information on the geometry of V. Let us just give a sample:

Theorem 2 *Let $V_3 \subset \mathbb{P}^4$ be a smooth cubic threefold. The divisor $\Theta \subset JV_3$ has a unique singular point p, which is a triple point. The tangent cone $\mathbb{P}T_p(\Theta) \subset \mathbb{P}T_p(JV_3) \cong \mathbb{P}^4$ is isomorphic to V_3.*

This elegant result, apparently due to Mumford (see [B2] for a proof), implies both the non-rationality of V_3 (because $\dim \mathrm{Sing}(\Theta) = 0$ and $\dim JV_3 = 5$) and the *Torelli theorem*: the cubic V_3 can be recovered from its (polarized) intermediate Jacobian.

There are actually few cases where we can control so well the singular locus of the Theta divisor. One of these is the quartic double solid, for which $\mathrm{Sing}(\Theta)$ has a component of codimension 5 in JV [V1]. Another case is that of *conic bundles*, that is, threefolds V with a flat morphism $p : V \to \mathbb{P}^2$, such that for each closed point $s \in \mathbb{P}^2$ the fiber $p^{-1}(s)$ is isomorphic to a plane conic (possibly singular). In that case JV is a *Prym variety*, associated to a natural double covering of the *discriminant curve* $\Delta \subset \mathbb{P}^2$ (the locus of $s \in \mathbb{P}^2$ such that $p^{-1}(s)$ is singular). Thanks to Mumford we have some control on the singularities of the Theta divisor of a Prym variety, enough to show that *JV is not a Jacobian (or a product of Jacobians) if* $\deg(\Delta) \geq 6$ [B1, Theorem 4.9].

Unfortunately, apart from the cubic, the only prime Fano threefold to which this result applies is the $V_{2,2,2}$ in $\mathbb{P}^6$. However, the Clemens-Griffiths criterion of non-rationality is an *open* condition. In fact, we have a stronger result, which follows from the properties of the *Satake compactification* of $\mathscr{A}_g$ [B1, Lemme 5.6.1]:

Lemma 3 *Let $\pi : V \to B$ be a flat family of projective threefolds over a smooth curve B. Let $\mathrm{o} \in B$; assume that:*

- *The fiber V_b is smooth for $b \neq \mathrm{o}$;*
- *V_{o} has only ordinary double points;*
- *For a desingularization $\tilde{V}_{\mathrm{o}}$ of V_{o}, $J\tilde{V}_{\mathrm{o}}$ is not a Jacobian or a product of Jacobians.*

Then for b outside a finite subset of B, V_b is not rational.

From this we deduce the generic non-rationality statements of Sect. 2.3 [B1, Theorem 5.6] : in each case one finds a degeneration as in the Lemma, such that $\tilde{V}_{\mathrm{o}}$ is a conic bundle with a discriminant curve of degree ≥ 6, hence the Lemma applies.

3.3 An Easy Counter-Example

The results of the previous section require rather involved methods. We will now discuss a much more elementary approach, which unfortunately applies only to specific varieties.

Theorem 3 *The cubic threefold* $V \subset \mathbb{P}^4$ *defined by* $\sum_{i\in\mathbb{Z}/5} X_i^2 X_{i+1} = 0$ *is not rational.*

Proof Let us first prove that JV is not a Jacobian. Let ζ be a primitive 11th root of unity. The key point is that V admits the automorphisms

$$\delta : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_0, \zeta X_1, \zeta^{-1} X_2, \zeta^3 X_3, \zeta^6 X_4),$$
$$\sigma : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_1, X_2, X_3, X_4, X_0),$$

which satisfy $\delta^{11} = \sigma^5 = 1$ and $\sigma\delta\sigma^{-1} = \delta^{-2}$.

They induce automorphisms δ^*, σ^* of JV. Suppose that JV is isomorphic (as p.p.a.v.) to the Jacobian JC of a curve C. The Torelli theorem for curves gives an exact sequence

$$1 \to \mathrm{Aut}(C) \to \mathrm{Aut}(JC) \to \mathbb{Z}/2\,;$$

since δ^* and σ^* have odd order, they are induced by automorphisms δ_C, σ_C of C, satisfying $\sigma_C\delta_C\sigma_C^{-1} = \delta_C^{-2}$.

Now we apply the Lefschetz fixed point formula. The automorphism δ of V fixes the five points corresponding to the basis vectors of $\mathbb{C}^5$; it acts trivially on $H^{2i}(V, \mathbb{Q})$ for $i = 0, \ldots, 3$. Therefore we find Tr $\delta^*_{|H^3(V,\mathbb{Q})} = -5 + 4 = -1$. Similarly σ fixes the four points $(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$ of V with $\alpha^5 = 1$, $\alpha \neq 1$, so Tr $\sigma^*_{|H^3(V,\mathbb{Q})} = -4 + 4 = 0$.

Applying now the Lefschetz formula to C, we find that σ_C has two fixed points on C and δ_C three. But since σ_C normalizes the subgroup generated by δ_C, it preserves the 3-points set $\mathrm{Fix}(\delta_C)$; since it is of order 5, it must fix each of these three points, which gives a contradiction.

Finally suppose JV is isomorphic to a product $A_1 \times \ldots \times A_p$ of p.p.a.v. By the unicity lemma (Lemma 2), the automorphism δ^* permutes the factors A_i. Since δ has order 11 and $p \leq 5$, this permutation must be trivial, so δ^* induces an automorphism of A_i for each i, hence of $H^1(A_i, \mathbb{Q})$; but the group $\mathbb{Z}/11$ has only one nontrivial irreducible representation defined over $\mathbb{Q}$, given by the cyclotomic field $\mathbb{Q}(\zeta)$, with $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 10$. Since $\dim(A_i) < 5$ we see that the action of δ^* on each A_i, and therefore on JV, is trivial. But this contradicts the relation Tr $\delta^*_{|H^3(V,\mathbb{Q})} = -1$. □

Remarks

1) The cubic V is the *Klein cubic threefold*; it is birational to the moduli space of abelian surfaces with a polarization of type $(1, 11)$ [G-P]. In particular it admits an action of the group $\mathrm{PSL}_2(\mathbb{F}_{11})$ of order 660, which is in fact its automorphism group [A]. From this one could immediately conclude by using the Hurwitz bound $\#\,\mathrm{Aut}(C) \leq 84(g(C) - 1)$ (see [B4]).
2) This method applies to other threefolds for which the non-rationality was not previously known, in particular the $\mathfrak{S}_7$-symmetric $V_{2,3}$ given by $\sum X_i = \sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^6$ [B4] or the $\mathfrak{S}_6$-symmetric V_4 with 30 nodes given by $\sum X_i = \sum X_i^4 = 0$ in $\mathbb{P}^5$ [B6].

4 Two Other Methods

In this section we will briefly present two other ways to get non-rationality results for certain Fano varieties. Let us stress that in dimension ≥ 4 these varieties are not known to be unirational, so these methods do not give us new counter-examples to the Lüroth problem.

4.1 Birational Rigidity

As mentioned in the introduction, Iskovskikh and Manin proved that a smooth quartic threefold $V_4 \subset \mathbb{P}^4$ is not rational by proving that any birational automorphism of V_4 is actually biregular. But they proved much more, namely that V_4 is *birationally superrigid* in the following sense:

Definition 3 Let V be a prime Fano variety (Sect. 2.3). We say that V is *birationally rigid* if:

a) There is no rational dominant map $V \dashrightarrow S$ with $0 < \dim(S) < \dim(V)$ and with general fibers of Kodaira dimension $-\infty$;
b) If V is birational to another prime Fano variety W, then V is isomorphic to W.

We say that V is *birationally superrigid* if any birational map $V \overset{\sim}{\dashrightarrow} W$ as in *b*) is an isomorphism.

(The variety W in *b*) is allowed to have certain mild singularities, the so-called $\mathbb{Q}$-factorial terminal singularities.)

After the pioneering work [I-M], birational (super)rigidity has been proved for a number of Fano varieties of index 1. Here is a sample; we refer to the surveys [P] and [Ch] for ideas of proofs and for many more examples.

- Any smooth hypersurface of degree n in $\mathbb{P}^n$ is birationally superrigid [dF].
- A general $V_{2,3}$ in $\mathbb{P}^5$ is birationally rigid. It is not birationally superrigid, since it contains a curve of lines, and each line defines by projection a 2-to-1 map to $\mathbb{P}^3$, hence a birational involution of $V_{2,3}$.
- A general $V_{d_1,\ldots,d_c}$ in $\mathbb{P}^n$ of index 1 (that is, $\sum d_i = n$) with $n > 3c$ is birationally superrigid.
- A double cover of $\mathbb{P}^n$ branched along a smooth hypersurface of degree $2n$ is birationally superrigid.

4.2 Reduction to Characteristic p

Theorem 4 ([K]) *For $d \geq 2\lceil \frac{n+3}{3} \rceil$, a very general hypersurface $V_d \subset \mathbb{P}^{n+1}$ is not ruled, and in particular not rational.*

A variety is *ruled* if it is birational to $W \times \mathbb{P}^1$ for some variety W. "Very general" means that the corresponding point in the space parametrizing our hypersurfaces lies outside a countable union of strict closed subvarieties.

The bound $d \geq 2\lceil \frac{n+3}{3} \rceil$ has been lowered to $d \geq 2\lceil \frac{n+2}{3} \rceil$ by Totaro [T]; this implies in particular that a very general quartic fourfold is not rational. More important, by combining Kollár's method with a new idea of Voisin (see Sect. 7), Totaro shows that a very general $V_d \subset \mathbb{P}^{n+1}$ with d as above is not *stably rational* (Sect. 5).

Let us give a very rough idea of Kollár's proof, in the case d is even. It starts from the well-known fact that the hypersurface V_d specializes to a double covering Y of a hypersurface of degree $d/2$. This can be still done in characteristic 2, at the price of getting some singularities on Y, which must be resolved. The reward is that the resolution Y' of Y has a very unstable tangent bundle; more precisely, $\Omega^{n-1}_{Y'}$ ($\cong T_{Y'} \otimes K_{Y'}$) contains a positive line bundle, and this prevents Y' to be ruled. Then a general result of Matsusaka implies that a very general V_d cannot be ruled.

5 Stable Rationality

There is an intermediate notion between rationality and unirationality which turns out to be important:

Definition 4 A variety V is *stably rational* if $V \times \mathbb{P}^n$ is rational for some $n \geq 0$.

In terms of field theory, this means that $\mathbb{C}(V)(t_1, \ldots, t_n)$ is a purely transcendantal extension of $\mathbb{C}$.

Clearly, rational $\Rightarrow$ stably rational $\Rightarrow$ unirational. We will see that these implications are strict. For the first one, we have:

Theorem 5 ([BCSS]) *Let $P(x,t) = x^3 + p(t)x + q(t)$ be an irreducible polynomial in $\mathbb{C}[x,t]$, whose discriminant $\delta(t) := 4p(t)^3 + 27q(t)^2$ has degree ≥ 5. The affine hypersurface $V \subset \mathbb{C}^4$ defined by $y^2 - \delta(t)z^2 = P(x,t)$ is stably rational but not rational.*

This answered a question asked by Zariski in 1949 [Sg1].

The non-rationality of V is proved using the intermediate Jacobian, which turns out to be the Prym variety associated to an admissible double covering of nodal curves. The stable rationality, more precisely the fact that $V \times \mathbb{P}^3$ is rational, was proved in [BCSS] using some particular torsors under certain algebraic tori. A slightly different approach due to Shepherd-Barron shows that actually $V \times \mathbb{P}^2$ is rational [SB]; we do not know whether $V \times \mathbb{P}^1$ is rational.

To find unirational varieties which are not stably rational, we cannot use the Clemens-Griffiths criterion since it applies only in dimension 3. The group of birational automorphisms is very complicated for a variety of the form $V \times \mathbb{P}^n$; so the only available method is the torsion of $H^3(V, \mathbb{Z})$ and its subsequent refinements, which we will examine in the next sections.

Remark There are other notions lying between unirationality and rationality. Let us say that a variety V is

- retract rational if there exists a rational dominant map $\mathbb{P}^N \dashrightarrow V$ which admits a rational section;
- factor-rational if there exists another variety V' such that $V \times V'$ is rational.

We have the implications:

rational $\Rightarrow$ stably rational $\Rightarrow$ factor-rational $\Rightarrow$ retract rational $\Rightarrow$ unirational.

Unfortunately at the moment we have no examples (even conjectural) of varieties which are retract rational but not stably rational. For this reason we will focus on the stable rationality, which seems at this time the most useful of these notions. Indeed we will see now that there are some classes of linear quotients V/G (see Sect. 2.4) for which we can prove stable rationality.

Let G be a reductive group acting on a variety V. We say that the action is *almost free* if there is a nonempty Zariski open subset U of V such that the stabilizer of each point of U is trivial.

Proposition 1 *Suppose that there exists an almost free linear representation V of G such that the quotient V/G is rational. Then for every almost free representation W of G, the quotient W/G is stably rational.*

The proof goes as follows [D]: let V^{o} be a Zariski open subset of V where G acts freely. Consider the diagonal action of G on $V^{\mathrm{o}} \times W$; standard arguments (the "no-name lemma") show that the projection $(V^{\mathrm{o}} \times W)/G \to V^{\mathrm{o}}/G$ defines a vector bundle over V^{o}/G. Thus $(V \times W)/G$ is birational to $(V/G) \times W$ (which is a rational variety), and symmetrically to $V \times (W/G)$, so W/G is stably rational. □

For many groups it is easy to find an almost free representation with rational quotient: this is the case for instance for a subgroup G of GL_n such that the quotient $\mathrm{GL_n}/G$ is rational (use the linear action of GL_n on $\mathrm{M}_n(\mathbb{C})$ by multiplication). This applies to GL_n, SL_n, O_n ($\mathrm{GL}_n/\mathrm{O}_n$ is the space of non-degenerate quadratic forms), SO_n, Sp_n etc.

This gives many examples of stably rational varieties. For instance, the moduli space $\mathscr{H}_{d,n}$ of hypersurfaces of degree d in $\mathbb{P}^n$ (Sect. 2.4) is stably rational when $d \equiv 1 \bmod. (n+1)$: the standard representation ρ of GL_{n+1} on $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ is not almost free, but the representation $\rho \otimes \det^k$, with $k = \frac{1-d}{n+1}$, is almost free and gives the same quotient.

6 The Torsion of $H^3(V, \mathbb{Z})$ and the Brauer Group

6.1 Birational Invariance

Artin and Mumford used the following property of stably rational varieties:

Proposition 2 *Let V be a stably rational projective manifold. Then $H^3(V,\mathbb{Z})$ is torsion free.*

Proof The Künneth formula gives an isomorphism

$$H^3(V \times \mathbb{P}^m, \mathbb{Z}) \cong H^3(V,\mathbb{Z}) \,\oplus\, H^1(V,\mathbb{Z})\ ;$$

since $H^1(V,\mathbb{Z})$ is torsion free the torsion subgroups of $H^3(V,\mathbb{Z})$ and $H^3(V\times\mathbb{P}^m,\mathbb{Z})$ are isomorphic, hence replacing V by $V\times\mathbb{P}^m$ we may assume that V is rational. Let $\varphi : \mathbb{P}^n \overset{\sim}{\dashrightarrow} V$ be a birational map. As in the proof of the Clemens-Griffiths criterion, we have Hironaka's "little roof"

$$\begin{tikzcd} & P \arrow[dl, "b"'] \arrow[dr, "f"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & V \end{tikzcd}$$

where $b : P \to \mathbb{P}^n$ is a composition of blowing up of smooth subvarieties, and f is a birational *morphism*.

By Lemma 1, we have $H^3(P,\mathbb{Z}) \cong H^1(Y_1,\mathbb{Z})\oplus\ldots\oplus H^1(Y_p,\mathbb{Z})$, where $Y_1,\ldots,Y_p$ are the subvarieties successively blown up by b; therefore $H^3(P,\mathbb{Z})$ is torsion free. As in the proof of Theorem 1, $H^3(V,\mathbb{Z})$ is a direct summand of $H^3(P,\mathbb{Z})$, hence is also torsion free. □

We will indicate below (Sect. 6.4) another proof which does not use Hironaka's difficult theorem.

6.2 The Brauer Group

The torsion of $H^3(V,\mathbb{Z})$ is strongly related to the *Brauer group* of V. There is a huge literature on the Brauer group in algebraic geometry, starting with the three "exposés" by Grothendieck in [G]. We recall here the cohomological definition(s) of this group; we refer to [G] for the relation with Azumaya algebras.

Proposition 3 *Let V be a smooth variety. The following definitions are equivalent, and define the* Brauer group *of V:*

(i) $\mathrm{Br}(V) = \mathrm{Coker}\ c_1 : \mathrm{Pic}(V)\otimes\mathbb{Q}/\mathbb{Z} \to H^2(V,\mathbb{Q}/\mathbb{Z})$*;*
(ii) $\mathrm{Br}(V) = H^2_{\text{ét}}(V,\mathbb{G}_m)$ *(étale cohomology).*

Proof Let $n \in \mathbb{N}$. The exact sequence of étale sheaves $1 \to \mathbb{Z}/n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 1$ gives a cohomology exact sequence

$$0 \to \mathrm{Pic}(V) \otimes \mathbb{Z}/n \xrightarrow{c_1} H^2(V, \mathbb{Z}/n) \longrightarrow \mathrm{Br}(V) \xrightarrow{\times n} \mathrm{Br}(V) \; .$$

(Note that the étale cohomology $H^*_{\text{ét}}(V, \mathbb{Z}/n)$ is canonically isomorphic to the classical cohomology.)

Taking the direct limit with respect to n gives an exact sequence

$$0 \to \mathrm{Pic}(V) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{c_1} H^2(V, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tors}\,\mathrm{Br}(V) \to 0 \; ; \qquad (1)$$

it is not difficult to prove that $\mathrm{Br}(V)$ is a torsion group [G, II, Proposition 1.4], hence the equivalence of the definitions (i) and (ii). □

Remark If V is compact, the same argument shows that $\mathrm{Br}(V)$ is also isomorphic to the torsion subgroup of $H^2(V, \mathcal{O}_h^*)$, where $\mathcal{O}_h$ is the sheaf of holomorphic functions on V (for the classical topology).

Proposition 4 *There is a surjective homomorphism* $\mathrm{Br}(V) \to \mathrm{Tors}\, H^3(V, \mathbb{Z})$*, which is bijective if* $c_1 : \mathrm{Pic}(V) \to H^2(V, \mathbb{Z})$ *is surjective.*

The latter condition is satisfied in particular if V is projective and $H^2(V, \mathcal{O}_V) = 0$.

Proof The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives a cohomology exact sequence

$$0 \to H^2(V, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H^2(V, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tors}\, H^3(V, \mathbb{Z}) \to 0 \; .$$

Together with (1) we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(V)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "c_1"'] & H^2(V,\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Br}(V) \arrow[r] & 0 \\
0 \arrow[r] & H^2(V,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & H^2(V,\mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Tors}\,H^3(V,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

which implies the Proposition. □

We will now describe a geometric way to construct nontrivial elements of the Brauer group.

Definition 5 Let V be a complex variety. A $\mathbb{P}^m$*-bundle* over V is a smooth map $p : P \to V$ whose geometric fibers are isomorphic to $\mathbb{P}^m$.

An obvious example is the projective bundle $\mathbb{P}_V(E)$ associated to a vector bundle E of rank $m+1$ on V; we will actually be interested in those $\mathbb{P}^m$-bundles which are *not* projective.

It is not difficult to see that a $\mathbb{P}^m$-bundle is locally trivial for the étale topology. This implies that isomorphism classes of $\mathbb{P}^{n-1}$-bundles over V are parametrized by the étale cohomology set $H^1(V, PGL_n)$, where for an algebraic group G we denote by G the sheaf of local maps to G. The exact sequence of sheaves of groups

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

gives rise to a sequence of pointed sets

$$H^1(V, GL_n) \xrightarrow{\pi} H^1(V, PGL_n) \xrightarrow{\partial} H^2(V, \mathbb{G}_m)$$

which is exact in the sense that $\partial^{-1}(1) = \operatorname{Im} \pi$. Thus ∂ associates to each $\mathbb{P}^{n-1}$-bundle $p : P \to V$ a class in $H^2(V, \mathbb{G}_m)$, which is trivial if and only if p is a projective bundle. Moreover, by comparing with the exact sequence $0 \to \mathbb{Z}/n \to SL_n \to PGL_n \to 1$ we get a commutative diagram

$$\begin{tikzcd}
H^1(V,SL_n) \arrow[r] \arrow[d] & H^1(V,PGL_n) \arrow[r] \arrow[d, equal] & H^2(V,\mathbb{Z}/n) \arrow[d] \\
H^1(V,GL_n) \arrow[r] & H^1(V,PGL_n) \arrow[r, "\partial"] & H^2(V,\mathbb{G}_m)
\end{tikzcd}$$

which shows that the image of ∂ is contained in the n-torsion subgroup of $\mathrm{Br}(V)$.

6.3 *The Artin-Mumford Example*

The Artin-Mumford counter-example is a double cover of $\mathbb{P}^3$ branched along a *quartic symmetroid*, that is, a quartic surface defined by the vanishing of a symmetric determinant.

We start with a web Π of quadrics in $\mathbb{P}^3$; its elements are defined by quadratic forms $\lambda_0 q_0 + \ldots + \lambda_3 q_3$. We assume that the following properties hold (this is the case when Π is general enough):

(i) Π is base point free;
(ii) If a line in $\mathbb{P}^3$ is singular for a quadric of Π, it is not contained in another quadric of Π.

Let $\Delta \subset \Pi$ be the discriminant locus, corresponding to quadrics of rank ≤ 3. It is a quartic surface (defined by $\det(\sum \lambda_i q_i) = 0$); under our hypotheses, $\mathrm{Sing}(\Delta)$ consists of 10 ordinary double points, corresponding to quadrics of rank 2 (see for instance [Co]). Let $\pi : V' \to \Pi$ be the double covering branched along Δ. Again V' has ten ordinary double points; blowing up these points we obtain the Artin-Mumford threefold V.

Observe that a quadric $q \in \Pi$ has two systems of generatrices (= lines contained in q) if $q \in \Pi \smallsetminus \Delta$, and one if $q \in \Delta \smallsetminus \mathrm{Sing}(\Delta)$. Thus the smooth part V^{o} of V parametrizes pairs (q, λ), where $q \in \Pi$ and λ is a family of generatrices of q.

Theorem 6 *The threefold V is unirational but not stably rational.*

Proof Let $\mathbb{G}$ be the Grassmannian of lines in $\mathbb{P}^3$. A general line is contained in a unique quadric of Π, and in a unique system of generatrices of this quadric; this defines a dominant rational map $\gamma : \mathbb{G} \dashrightarrow V'$, thus V is unirational. We will deduce from Proposition 2 that V is not stably rational, by proving that $H^3(V, \mathbb{Z})$ contains an element of order 2. This is done by a direct calculation in [A-M] and, with a different method, in [B3]; here we will use a more elaborate approach based on the Brauer group.

Consider the variety $P \subset \mathbb{G} \times \Pi$ consisting of pairs (ℓ, q) with $\ell \subset q$. The projection $P \to \Pi$ factors through a morphism $p' : P \to V'$. Put $V^{\mathrm{o}} := V' \smallsetminus \mathrm{Sing}(V')$, and $P^{\mathrm{o}} := p'^{-1}(V^{\mathrm{o}})$. The restriction $p : P^{\mathrm{o}} \to V^{\mathrm{o}}$ is a $\mathbb{P}^1$-bundle: a point of V^{o} is a pair (q, σ), where q is a quadric in Π and σ a system of generatrices of q; the fiber $p^{-1}(q, \sigma)$ is the smooth rational curve parametrizing the lines of σ.

Proposition 5 *The $\mathbb{P}^1$-bundle $p : P^{\mathrm{o}} \to V^{\mathrm{o}}$ does not admit a rational section.*

Proof Suppose it does. For a general point q of Π, the section maps the two points of $\pi^{-1}(q)$ to two generatrices of the quadric q, one in each system. These two generatrices intersect in one point $s(q)$ of q; the map $q \mapsto s(q)$ is a rational section of the universal family of quadrics $\mathscr{Q} \to \Pi$, defined by $\mathscr{Q} := \{(q, x) \in \Pi \times \mathbb{P}^3 \mid x \in q\}$. This contradicts the following lemma:

Lemma 4 *Let $\Pi \subset \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ be a base point free linear system of hypersurfaces, of degree $d \geq 2$. Consider the universal family $p : \mathscr{H} \to \Pi$, with $\mathscr{H} := \{(h, x) \in \Pi \times \mathbb{P}^n \mid x \in h\}$. Then p has no rational section.*

Proof Since Π is base point free, the second projection $q : \mathscr{H} \to \mathbb{P}^n$ is a projective bundle, hence $\mathscr{H}$ is smooth. If p has a rational section, the closure $Z \subset \mathscr{H}$ of its image gives a cohomology class $[Z] \in H^{2n-2}(\mathscr{H}, \mathbb{Z})$ such that $p_*([Z]) = 1$ in $H^0(\Pi, \mathbb{Z})$. Let us show that this is impossible.

We have $\dim(\Pi) \geq n$, hence $2n - 2 < n - 1 + \dim(\Pi) = \dim(\mathscr{H})$. By the Lefschetz hyperplane theorem, the restriction map $H^{2n-2}(\Pi \times \mathbb{P}^n, \mathbb{Z}) \to H^{2n-2}(\mathscr{H}, \mathbb{Z})$ is an isomorphism. Thus $H^{2n-2}(\mathscr{H}, \mathbb{Z})$ is spanned by the classes $p^*h_\Pi^i \cdot q^*h_{\mathbb{P}}^{n-1-i}$ for $0 \leq i \leq n-1$, where h_Π and $h_{\mathbb{P}}$ are the hyperplane classes. All these classes go to 0 under p_* except $q^*h_{\mathbb{P}}^{n-1}$, whose degree on each fiber is d. Thus the image of $p_* : H^{2n-2}(\mathscr{H}, \mathbb{Z}) \to H^0(\Pi, \mathbb{Z}) = \mathbb{Z}$ is $d\mathbb{Z}$. This proves the lemma, hence the Proposition. ◻

Thus the $\mathbb{P}^1$-bundle p over V^o is not a projective bundle, hence gives a nonzero 2-torsion class in $\mathrm{Br}(V^o)$. In the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(V) \arrow[r, "c_1"] \arrow[d] & H^2(V,\mathbb{Z}) \arrow[d, "r"] \\
\mathrm{Pic}(V^o) \arrow[r, "c_1"] & H^2(V^o,\mathbb{Z})
\end{tikzcd}$$

the top horizontal arrow is surjective because $H^2(V,\mathscr{O}_V) = 0$. Since $Q := V \smallsetminus V^o$ is a disjoint union of quadrics, the Gysin exact sequence

$$H^2(V,\mathbb{Z}) \xrightarrow{r} H^2(V^o,\mathbb{Z}) \to H^1(Q,\mathbb{Z}) = 0$$

shows that r is surjective. Therefore the map $c_1 : \mathrm{Pic}(V^o) \to H^2(V^o,\mathbb{Z})$ is surjective, and by Proposition 4 we get a nonzero 2-torsion class in $H^3(V^o,\mathbb{Z})$. Using again the Gysin exact sequence

$$0 \to H^3(V,\mathbb{Z}) \to H^3(V^o,\mathbb{Z}) \to H^2(Q,\mathbb{Z})$$

we find that $\mathrm{Tors}\, H^3(V,\mathbb{Z})$ is isomorphic to $\mathrm{Tors}\, H^3(V^o,\mathbb{Z})$, hence nonzero. □

6.4 The Unramified Brauer Group

An advantage of the group $\mathrm{Br}(V)$ is that it can be identified with the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V))$, which is defined purely in terms of the field $\mathbb{C}(V)$; this gives directly its birational invariance, without using Hironaka's theorem. Let us explain briefly how this works.

Proposition 6 *Let V be a smooth projective variety, and $\mathscr{D}$ be the set of integral divisors on V. There is an exact sequence*

$$0 \to \mathrm{Br}(V) \to \varinjlim_{U} \mathrm{Br}(U) \to \bigoplus_{D\in\mathscr{D}} H^1_{\text{ét}}(\mathbb{C}(D),\mathbb{Q}/\mathbb{Z})$$

where the direct limit is taken over the set of Zariski open subsets $U \subset V$.

Proof Let D be an effective reduced divisor on V, and let $U = V \smallsetminus D$. Since $\mathrm{Sing}(D)$ has codimension ≥ 2 in V, the restriction map

$$H^2(V,\mathbb{Q}/\mathbb{Z}) \to H^2(V \smallsetminus \mathrm{Sing}(D),\mathbb{Q}/\mathbb{Z})$$

is an isomorphism. Thus, putting $D_{sm} := D \smallsetminus \mathrm{Sing}(D)$, we can write part of the Gysin exact sequence as

$$H^0(D_{sm}, \mathbb{Q}/\mathbb{Z}) \to H^2(V, \mathbb{Q}/\mathbb{Z}) \to H^2(U, \mathbb{Q}/\mathbb{Z}) \to H^1(D_{sm}, \mathbb{Q}/\mathbb{Z}) .$$

Comparing with the analogous exact sequence for Picard groups gives a commutative diagram

$$\begin{tikzcd}
H^0(D_{sm}, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\wr"] & \mathrm{Pic}(V) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & \mathrm{Pic}(U) \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
H^0(D_{sm}, \mathbb{Q}/\mathbb{Z}) \arrow[r] & H^2(V, \mathbb{Q}/\mathbb{Z}) \arrow[r] & H^2(U, \mathbb{Q}/\mathbb{Z}) \arrow[r] & H^1(D_{sm}, \mathbb{Q}/\mathbb{Z})
\end{tikzcd}$$

from which we get an exact sequence $0 \to \mathrm{Br}(V) \to \mathrm{Br}(U) \to H^1(D_{sm}, \mathbb{Q}/\mathbb{Z})$.

Let $D_1, \ldots, D_k$ be the irreducible components of D_{sm}; we have $H^1(D_{sm}, \mathbb{Q}/\mathbb{Z}) = \oplus H^1(D_{sm} \cap D_i, \mathbb{Q}/\mathbb{Z})$, and the group $H^1(D_{sm} \cap D_i, \mathbb{Q}/\mathbb{Z})$ embeds into the étale cohomology group $H^1_{\text{ét}}(\mathbb{C}(D_i), \mathbb{Q}/\mathbb{Z})$. Thus we can write our exact sequence

$$0 \to \mathrm{Br}(V) \to \mathrm{Br}(U) \to \bigoplus_i H^1_{\text{ét}}(\mathbb{C}(D_i), \mathbb{Q}/\mathbb{Z}) .$$

Passing to the limit over D gives the Proposition. □

Let K be a field. For each discrete valuation ring (DVR) R with quotient field K and residue field κ_R, there is a natural exact sequence [G, III, Proposition 2.1]:

$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \xrightarrow{\rho_R} H^1_{\text{ét}}(\kappa_R, \mathbb{Q}/\mathbb{Z}) .$$

The group $\mathrm{Br}_{\mathrm{nr}}(K)$ is defined as the intersection of the subgroups $\mathrm{Ker}\ \rho_R$, where R runs through all DVR with quotient field K.

Now consider the exact sequence of Proposition 6. The group $\varinjlim_U \mathrm{Br}(U)$ can be identified with the Brauer group $\mathrm{Br}(\mathbb{C}(V))$, and the homomorphism $\mathrm{Br}(\mathbb{C}(V)) \to H^1_{\text{ét}}(\mathbb{C}(D), \mathbb{Q}/\mathbb{Z})$ coincides with the homomorphism $\rho_{\mathscr{O}_{V,D}}$ associated to the DVR $\mathscr{O}_{V,D}$. Thus we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)) \subset \mathrm{Br}(V)$. But if R is any DVR with quotient field $\mathbb{C}(V)$, the inclusion $\mathrm{Spec}\,\mathbb{C}(V) \hookrightarrow V$ factors as $\mathrm{Spec}\,\mathbb{C}(V) \hookrightarrow \mathrm{Spec}\,R \to V$ by the valuative criterion of properness, hence $\mathrm{Br}(V)$ is contained in the image of $\mathrm{Br}(R)$ in $\mathrm{Br}(K)$, that is, in $\mathrm{Ker}\ \rho_R$. Thus we have $\mathrm{Br}(V) = \mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V))$ as claimed.

The big advantage of working with $\mathrm{Br}_{\mathrm{nr}}(K)$ is that to compute it, we do not need to find a smooth projective model of the function field K. This was used first by Saltman to give his celebrated counter-example to the Noether problem [Sa]: there exists a finite group G and a linear representation V of G such that the variety V/G is not rational. In such a situation Bogomolov has given a very explicit formula for $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V/G))$ in terms of the Schur multiplier of G [Bo].

The idea of using the unramified Brauer group to prove non-rationality results has been extended to higher *unramified cohomology groups*, starting with the paper [C-O]. We refer to [C] for a survey about these more general invariants.

7 The Chow Group of 0-Cycles

In this section we discuss another property of (stably) rational varieties, namely the fact that their Chow group CH_0 parametrizing 0-cycles is *universally trivial*. While the idea goes back to the end of the 1970s (see [Bl]), its use for rationality questions is recent [V4].

This property implies that $H^3(X,\mathbb{Z})$ is torsion free, but not conversely. Moreover it behaves well under deformation, even if we accept mild singularities (Proposition 9 below).

In this section we will need to work over non-algebraically closed fields (of characteristic 0). We use the language of schemes.

Let X be a smooth algebraic variety over a field k, of dimension n. Recall that the Chow group $CH^p(X)$ is the group of codimension p cycles on X modulo linear equivalence. More precisely, let us denote by $\Sigma^p(X)$ the set of codimension p closed integral subvarieties of X. Then $CH^p(X)$ is defined by the exact sequence

$$\bigoplus_{W\in\Sigma^{p-1}(X)} k(W)^* \longrightarrow \mathbb{Z}^{(\Sigma^p(X))} \longrightarrow CH^p(X) \to 0\,, \tag{2}$$

where the first arrow associates to $f \in k(W)^*$ its divisor [Fu, 1.3].

We will be particularly interested in the group $CH_0(X) := CH^n(X)$ of 0-cycles. Associating to a 0-cycle $\sum n_i[p_i]$ $(n_i \in \mathbb{Z}, p_i \in X)$ the number $\sum n_i[k(p_i):k]$ defines a homomorphism $\deg : CH_0(X) \to \mathbb{Z}$. We denote its kernel by $CH_0(X)_0$.

Proposition 7 *Let X be a smooth complex projective variety, of dimension n, and let $\Delta_X \subset X \times X$ be the diagonal. The following conditions are equivalent:*

(i) *For every extension $\mathbb{C} \to K$, $CH_0(X_K)_0 = 0$;*
(ii) *$CH_0(X_{\mathbb{C}(X)})_0 = 0$;*
(iii) *There exists a point $x \in X$ and a nonempty Zariski open subset $U \subset X$ such that $\Delta_X - X \times \{x\}$ restricts to 0 in $CH(U \times X)$;*
(iv) *there exists a point $x \in X$, a smooth projective variety T of dimension $< n$ (not necessarily connected), a generically injective map $i : T \to X$, and a cycle class $\alpha \in CH(T \times X)$ such that*

$$\Delta_X - X \times \{x\} = (i \times 1)_*\alpha \quad \textit{in} \;\; CH(X \times X)\,. \tag{3}$$

When these properties hold, we say that X is CH_0-trivial.

Proof The implication (i) ⇒ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let η be the generic point of X. The point (η,η) of $\{\eta\}\times X = X_{\mathbb{C}(X)}$ is rational (over $\mathbb{C}(X)$), hence is linearly equivalent to (η,x) for any closed point $x\in X$. The class $\Delta_X - X\times\{x\}$ restricts to $(\eta,\eta)-(\eta,x)$ in $CH_0(\eta\times X)$, hence to 0. We want to show that this implies (iii).

An element of $\Sigma^p(\eta\times X)$ extends to an element of $\Sigma^p(U\times X)$ for some Zariski open subset U of X; in other words, the natural map $\varinjlim_U \Sigma^p(U\times X)\to\Sigma^p(\eta\times X)$ is an isomorphism. Thus writing down the exact sequence (2) for $U\times X$ and passing to the direct limit over U we get a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\varinjlim_U \bigoplus_{W\in\Sigma^{p-1}(U\times X)} k(W)^* & \longrightarrow & \varinjlim_U \mathbb{Z}^{(\Sigma^p(U\times X))} & \longrightarrow & \varinjlim_U CH^p(U\times X) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{W\in\Sigma^{p-1}(\eta\times X)} k(W)^* & \longrightarrow & \mathbb{Z}^{(\Sigma^p(\eta\times X))} & \longrightarrow & CH^p(\eta\times X) & \longrightarrow & 0
\end{array}$$

where the first two vertical arrows are isomorphisms; therefore the third one is also an isomorphism. We conclude that the class $\Delta - X\times\{x\}$ is zero in $CH^n(U\times X)$ for some U.

(iii) $\Rightarrow$ (iv): Put $T' := X\smallsetminus U$. The localization exact sequence [Fu, Prop. 1.8]

$$CH(T'\times X)\to CH(X\times X)\to CH(U\times X)\to 0$$

implies that $\Delta - X\times\{x\}$ comes from the class in $CH(T'\times X)$ of a cycle $\sum n_i Z_i'$. For each i, let T_i' be the image of Z_i in T', and let T_i be a desingularization of T_i'. Since Z_i' is not contained in the singular locus $\mathrm{Sing}(T_i')\times X$, it is the pushforward of an irreducible subvariety $Z_i\subset T_i\times X$. Putting $T=\coprod T_i$ and $\alpha=\sum n_i[Z_i]$ does the job.

(iv) $\Rightarrow$ (i): Assume that (3) holds; then it holds in $CH(X_K\times X_K)$ for any extension K of $\mathbb{C}$, so it suffices to prove $CH_0(X)_0=0$.

Denote by p and q the two projections from $X\times X$ to X, and put $n:=\dim(X)$. Any class $\delta\in CH^n(X\times X)$ induces a homomorphism $\delta_*: CH_0(X)\to CH_0(X)$, defined by $\delta_*(z)=q_*(\delta\cdot p^*z)$. Let us consider the classes which appear in (3). The diagonal induces the identity of $CH_0(X)$; the class of $X\times\{x\}$ maps $z\in CH_0(X)$ to $\deg(z)\,[x]$, hence is 0 on $CH_0(X)_0$.

Now consider $\delta:=(i\times 1)_*\alpha$. Let p',q' be the projections from $T\times X$ to T and X. Then, for $z\in CH_0(X)$,

$$\delta_* z = q_*((i\times 1)_*\alpha\cdot p^*z) = q'_*(\alpha\cdot p'^*i^*z)\ .$$

Since $\dim T<\dim X$, i^*z is zero, hence also δ_*z. We conclude from (3) that $CH_0(X)_0=0$. $\square$

Example The group $CH_0(X)$ is a birational invariant [Fu, ex. 16.1.11], thus the above properties depend only on the birational equivalence class of X. In particular a rational variety is CH_0-trivial. More generally, since $CH_0(X \times \mathbb{P}^n) \cong CH_0(X)$ for any variety X, a stably rational variety is CH_0-trivial.

Despite its technical aspect, Proposition 7 has remarkable consequences (see e.g. [B-S]):

Proposition 8 *Suppose X is CH_0-trivial.*

1) $H^0(X, \Omega_X^r) = 0$ *for all* $r > 0$.
2) *The group* $H^3(X, \mathbb{Z})$ *is torsion free.*

Proof The proof is very similar to that of the implication (iv) ⇒ (i) in the previous Proposition; we use the same notation. Again a class δ in $CH^n(X \times X)$ induces a homomorphism $\delta^* : H^r(X, \mathbb{Z}) \to H^r(X, \mathbb{Z})$, defined by $\delta^* z := p_*(\delta \cdot q^* z)$. The diagonal induces the identity, the class $[X \times \{p\}]$ gives 0 for $r > 0$, and the class $(i \times 1)_* \alpha$ gives the homomorphism $z \mapsto i_* p'_*(\alpha \cdot q'^* z)$. Thus formula (3) gives for $r > 0$ a commutative diagram

$$\begin{tikzcd} & H^*(T, \mathbb{Z}) \arrow[dr, "i_*"] & \\ H^r(X, \mathbb{Z}) \arrow[ur] \arrow[rr, "\mathrm{Id}"] & & H^r(X, \mathbb{Z}) . \end{tikzcd} \tag{4}$$

On each component T_k of T the homomorphism $i_* : H^*(T_k, \mathbb{C}) \to H^*(X, \mathbb{C})$ is a morphism of Hodge structures of bidegree (c, c), with $c := \dim X - \dim T_k > 0$. Therefore its image intersects trivially the subspace $H^{r,0}$ of $H^r(X, \mathbb{C})$. Since i_* is surjective by (4), we get $H^{r,0} = 0$.

Now we take $r = 3$ in (4). The only part of $H^*(T, \mathbb{Z})$ with a nontrivial contribution in (4) is $H^1(T, \mathbb{Z})$, which is torsion free. Any torsion element in $H^3(X, \mathbb{Z})$ goes to 0 in $H^1(T, \mathbb{Z})$, hence is zero. □

Observe that in the proof we use only formula (3) in $H^*(X \times X)$ and not in the Chow group. The relation between these two properties is discussed in Voisin's papers [V3, V4, V5].

As the Clemens-Griffiths criterion, the triviality of $CH_0(X)$ behaves well under deformation (compare with Lemma 3):

Proposition 9 ([V4]) *Let $\pi : X \to B$ be a flat, proper family over a smooth variety B, with* $\dim(X) \geq 3$. *Let* $o \in B$*; assume that:*

- *The general fiber X_b is smooth;*
- *X_o has only ordinary double points, and its desingularization $\tilde{X}_o$ is not CH_0-trivial.*

Then X_b is not CH_0-trivial for a very general point b of B.

Recall that 'very general' means 'outside a countable union of strict subvarieties of B' (Sect. 4.2).

We refer to [V4] for the proof. The idea is that there cannot exist a decomposition (3) of Proposition 7 for b general in B, because it would extend to an analogous decomposition over X, then specialize to X_o, and finally extend to $\tilde{X}_o$. One concludes by observing that the locus of points $b \in B$ such that X_b is smooth and CH_0-trivial is a countable union of subvarieties.

Corollary 1 *The double cover of $\mathbb{P}^3$ branched along a very general quartic surface is not stably rational.*

Proof Consider the pencil of quartic surfaces in $\mathbb{P}^3$ spanned by a smooth quartic and a quartic symmetroid, and the family of double covers of $\mathbb{P}^3$ branched along the members of this pencil. By Proposition 8.2), the Artin-Mumford threefold is not CH_0-trivial. Applying the Proposition we conclude that a very general quartic double solid is not CH_0-trivial, hence not stably rational. □

More generally, Voisin shows that the desingularization of a very general quartic double solid with at most seven nodes is not stably rational.

Voisin's idea has given rise to a number of new results. Colliot-Thélène and Pirutka have extended Proposition 9 to the case where the singular fiber X_o has (sufficiently nice) non-isolated singularities, and applied this to prove that a very general quartic threefold is not stably rational [C-P1]. I proved that a very general sextic double solid is not stably rational [B7]. As already mentioned, combining the methods of Kollár and Colliot-Thélène-Pirutka, Totaro has proved that a very general hypersurface of degree d and dimension n is not stably rational for $d \geq 2\lceil\frac{n+2}{3}\rceil$ [T]; Colliot-Thélène and Pirutka have extended this to certain cyclic coverings [C-P2]. Hassett, Kresch and Tschinkel have shown that a conic bundle (see Sect. 3.2) with discriminant a very general plane curve of degree ≥ 6 is not stably rational [HKT]. Finally, using the result of [HKT], Hassett and Tschinkel have proved that a very general Fano threefold which is not rational or birational to a cubic threefold is not stably rational [HT].

We do not know whether there exist smooth quartic double solids which are CH_0-trivial. In contrast, Voisin has constructed families of smooth cubic threefolds which are CH_0-trivial [V5]—we do not know what happens for a general cubic threefold.

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Cubic Fourfolds, K3 Surfaces, and Rationality Questions

Brendan Hassett

Abstract This is a survey of the geometry of complex cubic fourfolds with a view toward rationality questions. Topics include classical constructions of rational examples, Hodge structures and special cubic fourfolds, associated K3 surfaces and their geometric interpretations, and connections with rationality and unirationality constructions.

This is a survey of the geometry of complex cubic fourfolds with a view toward rationality questions. Smooth cubic surfaces have been known to be rational since the nineteenth century [Dol05]; cubic threefolds are irrational by the work of Clemens and Griffiths [CG72]. Cubic fourfolds are likely more varied in their behavior. While there are examples known to be rational, we expect that most cubic fourfolds should be irrational. However, no cubic fourfolds are *proven* to be irrational.

Our organizing principle is that progress is likely to be driven by the dialectic between concrete geometric constructions (of rational, stably rational, and unirational parametrizations) and conceptual tools differentiating various classes of cubic fourfolds (Hodge theory, moduli spaces and derived categories, and decompositions of the diagonal). Thus the first section of this paper is devoted to classical examples of rational parametrizations. In Sect. 2 we focus on Hodge-theoretic classifications of cubic fourfolds with various special geometric structures. These are explained in Sect. 3 using techniques from moduli theory informed by deep results on K3 surfaces and their derived categories. We return to constructions in the fourth section, focusing on unirational parametrizations of special classes of cubic fourfolds. In the last section, we touch on recent applications of decompositions of the diagonal to rationality questions, and what they mean for cubic fourfolds.

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1 Introduction and Classical Constructions

Throughout, we work over the complex numbers.

1.1 Basic Definitions

Let X be a smooth projective variety of dimension n. If there exists a birational map $\rho : \mathbb{P}^n \overset{\sim}{\dashrightarrow} X$ we say that X is *rational*. It is *stably rational* if $X \times \mathbb{P}^m$ is rational for some $m \geq 0$. If there exists a generically finite map $\rho : \mathbb{P}^n \dashrightarrow X$ we say that X is *unirational*; this is equivalent to the existence of a dominant map from a projective space of arbitrary dimension to X.

A *cubic fourfold* is a smooth cubic hypersurface $X \subset \mathbb{P}^5$, with defining equation

$$F(u, v, w, x, y, z) = 0$$

where $F \in \mathbb{C}[u, v, w, x, y, z]$ is homogeneous of degree three. Cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by

$$\mathbb{P}(\mathbb{C}[u, v, w, x, y, z]_3) \simeq \mathbb{P}^{55}$$

with the smooth ones corresponding to a Zariski open $U \subset \mathbb{P}^{55}$.

Sometimes we will consider singular cubic hypersurfaces; in these cases, we shall make explicit reference to the singularities. The singular cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by an irreducible divisor

$$\Delta := \mathbb{P}^{55} \setminus U.$$

Birationally, Δ is a $\mathbb{P}^{49}$ bundle over $\mathbb{P}^5$, as having a singularity at a point $p \in \mathbb{P}^5$ imposes six independent conditions.

The moduli space of cubic fourfolds is the quotient

$$\mathcal{C} := [U/\operatorname{PGL}_6].$$

This is a Deligne-Mumford stack with quasi-projective coarse moduli space, e.g., by classical results on the automorphisms and invariants of hypersurfaces [MFK94, Chap. 4.2]. Thus we have

$$\dim(\mathcal{C}) = \dim(U) - \dim(\operatorname{PGL}_6) = 55 - 35 = 20.$$

1.2 Cubic Fourfolds Containing Two Planes

Fix disjoint projective planes

$$P_1 = \{u = v = w = 0\}, P_2 = \{x = y = z = 0\} \subset \mathbb{P}^5$$

and consider the cubic fourfolds X containing P_1 and P_2. For a concrete equation, consider

$$X = \{ux^2 + vy^2 + wz^2 = u^2x + v^2y + w^2z\}$$

which is in fact smooth! See [HK07, Sect. 5] for more discussion of this example.

More generally, fix forms

$$F_1, F_2 \in \mathbb{C}[u, v, w; x, y, z]$$

of bidegree $(1, 2)$ and $(2, 1)$ in the variables $\{u, v, w\}$ and $\{x, y, z\}$. Then the cubic hypersurface

$$X = \{F_1 + F_2 = 0\} \subset \mathbb{P}^5$$

contains P_1 and P_2, and the defining equation of every such hypersurface takes that form. Up to scaling, these form a projective space of dimension 35. The group

$$\{g \in \mathrm{PGL}_6 : g(P_1) = P_1, g(P_2) = P_2\}$$

has dimension 17. Thus the locus of cubic fourfolds containing a pair of disjoint planes has codimension two in $\mathcal{C}$.

The cubic fourfolds of this type are *rational*. Indeed, we construct a birational map as follows: Given points $p_1 \in P_1$ and $p_2 \in P_2$, let $\ell(p_1, p_2)$ be the line containing them. The Bezout Theorem allows us to write

$$\ell(p_1, p_2) \cap X = \begin{cases} \{p_1, p_2, \rho(p_1, p_2)\} \text{ if } \ell(p_1, p_2) \not\subset X \\ \ell(p_1, p_2) \text{ otherwise.} \end{cases}$$

The condition $\ell(p_1, p_2) \subset X$ is expressed by the equations

$$S := \{F_1(u, v, w; x, y, z) = F_2(u, v, w; x, y, z) = 0\} \subset P_{1,[x,y,z]} \times P_{2,[u,v,w]}.$$

Since S is a complete intersection of hypersurfaces of bidegrees $(1, 2)$ and $(2, 1)$ it is a K3 surface, typically with Picard group of rank two. Thus we have a well-defined morphism

$$\begin{aligned} \rho : P_1 \times P_2 \setminus S &\to X \\ (p_1, p_2) &\mapsto \rho(p_1, p_2) \end{aligned}$$

that is birational, as each point of $\mathbb{P}^5 \setminus (P_1 \cup P_2)$ lies on a unique line joining the planes.

We record the linear series inducing this birational parametrization: ρ is given by the forms of bidegree $(2, 2)$ containing S and ρ^{-1} by the quadrics in $\mathbb{P}^5$ containing P_1 and P_2.

1.3 Cubic Fourfolds Containing a Plane and Odd Multisections

Let X be a cubic fourfold containing a plane P. Projection from P gives a quadric surface fibration

$$q : \tilde{X} := \mathrm{Bl}_P(X) \to \mathbb{P}^2$$

with singular fibers over a sextic curve $B \subset \mathbb{P}^2$. If q admits a rational section then $\tilde{X}$ is rational over $K = \mathbb{C}(\mathbb{P}^2)$ and thus over $\mathbb{C}$ as well. The simplest example of such a section is another plane disjoint from P. Another example was found by Tregub [Tre93]: Suppose there is a quartic Veronese surface

$$V \simeq \mathbb{P}^2 \subset X$$

meeting P transversally at three points. Then its proper transform $\tilde{V} \subset \tilde{X}$ is a section of q, giving rationality.

To generalize this, we employ a basic property of quadric surfaces due to Springer (cf. [Has99, Proposition 2.1] and [Swa89]):

> Let $Q \subset \mathbb{P}^3_K$ be a quadric surface smooth over a field K. Suppose there exists an extension L/K of odd degree such that $Q(L) \neq \emptyset$. Then $Q(K) \neq \emptyset$ and Q is rational over K via projection from a rational point.

This applies when there exists a surface $W \subset X$ intersecting the generic fiber of q transversally in an odd number of points. Thus we the following:

Theorem 1 ([Has99]) *Let X be a cubic fourfold containing a plane P and projective surface W such that*

$$\deg(W) - \langle P, W \rangle$$

is odd. Then X is rational.

The intersection form on the middle cohomology of X is denoted by $\langle , \rangle$.

Theorem 1 gives a countably infinite collection of codimension two subvarieties in $\mathcal{C}$ parametrizing rational cubic fourfolds. Explicit birational maps $\rho : \mathbb{P}^2 \times \mathbb{P}^2 \overset{\sim}{\dashrightarrow} X$ can be found in many cases [Has99, Sect. 5].

We elaborate the geometry behind Theorem 1: Consider the relative variety of lines of the quadric surface fibration q

$$f : F_1(\tilde{X}/\mathbb{P}^2) \to \mathbb{P}^2.$$

For each $p \in \mathbb{P}^2$, $f^{-1}(p)$ parametrizes the lines contained in the quadric surface $q^{-1}(p)$. When the fiber is smooth, this is a disjoint union of two smooth $\mathbb{P}^1$'s; for $p \in B$, we have a single $\mathbb{P}^1$ with multiplicity two. Thus the Stein factorization

$$f : F_1(\tilde{X}/\mathbb{P}^2) \to S \to \mathbb{P}^2$$

yields a degree two K3 surface—the double cover $S \to \mathbb{P}^2$ branched over B—and a $\mathbb{P}^1$-bundle $r : F_1(\tilde{X}/\mathbb{P}^2) \to S$. The key to the proof is the equivalence of the following conditions (see also [Kuz17, Theorem 4.11]):

- the generic fiber of q is rational over K;
- q admits a rational section;
- r admits a rational section.

The resulting birational map $\rho^{-1} : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ blows down the lines incident to the section of q, which are parametrized by a surface birational to S.

Cubic fourfolds containing a plane have been re-examined recently from the perspective of twisted K3 surfaces and their derived categories [Kuz10, MS12, Kuz17]. The twisted K3 surface is the pair (S, η), where η is the class in the Brauer group of S arising from r; note that $\eta = 0$ if and only if the three equivalent conditions above hold. Applications of this geometry to rational points may be found in [HVAV11].

Remark 2 The technique of Theorem 1 applies to all smooth projective fourfolds admitting quadric surface fibrations $Y \to P$ over a rational surface P. Having an odd multisection suffices to give rationality.

1.4 Cubic Fourfolds Containing Quartic Scrolls

A *quartic scroll* is a smooth rational ruled surface $\Sigma \hookrightarrow \mathbb{P}^5$ with degree four, with the rulings embedded as lines. There are two possibilities:

- $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the linear series $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)|$
- the Hirzebruch surface $\mathbb{F}_2$ embedded via $|\mathcal{O}_{\mathbb{F}_2}(\xi + f)|$ where f is a fiber and ξ a section at infinity ($f\xi = 1$ and $\xi^2 = 2$).

The second case is a specialization of the first. Note that all scrolls of the first class are projectively equivalent and have equations given by the 2×2 minors of:

$$\begin{pmatrix} u & v & x & y \\ v & w & y & z \end{pmatrix}$$

Lemma 3 *Let Σ be a quartic scroll, realized as the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the linear series $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)|$. Then a generic point $p \in \mathbb{P}^5$ lies on a unique secant to Σ. The locus of points on more than one secant equals the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2$ associated with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of the second factor.*

Proof The first assertion follows from a computation with the double point formula [Ful84, Sect. 9.3]. For the second, if two secants to Σ, $\ell(s_1, s_2)$ and $\ell(s_3, s_4)$, intersect then $s_1, \ldots, s_4$ are coplanar. But points $s_1, \ldots, s_4 \in \Sigma$ that fail to impose independent conditions on $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)|$ necessarily have at least three points on a line or all the points on a conic contained in Σ. □

Surfaces in $\mathbb{P}^5$ with 'one apparent double point' have been studied for a long time. See [Edg32] for discussion and additional classical references and [BRS15] for a modern application to cubic fourfolds.

Proposition 4 *If X is a cubic fourfold containing a quartic scroll Σ then X is rational.*

Here is the idea: Consider the linear series of quadrics cutting out Σ. It induces a morphism

$$\mathrm{Bl}_{\Sigma}(X) \to \mathbb{P}(\Gamma(\mathcal{I}_{\Sigma}(2))) \simeq \mathbb{P}^5,$$

mapping X birationally to a quadric hypersurface. Thus X is rational.

Remark 5 Here is another approach. Let $R \simeq \mathbb{P}^1$ denote the ruling of Σ; for $r \in R$, let $\ell(r) \subset \Sigma \subset X$ denote the corresponding line. For distinct $r_1, r_2 \in R$, the intersection

$$\mathrm{span}(\ell(r_1), \ell(r_2)) \cap X$$

is a cubic surface containing disjoint lines. Let Y denote the closure

$$\{(x, r_1, r_2) : x \in \mathrm{span}(\ell(r_1), \ell(r_2)) \cap X\} \subset X \times \mathrm{Sym}^2(R) \simeq X \times \mathbb{P}^2.$$

The induced $\pi_2 : Y \to \mathbb{P}^2$ is a cubic surface fibration such that the generic fiber contains two lines. Thus the generic fiber Y_K, $K = \mathbb{C}(\mathbb{P}^2)$, is rational over K and consequently Y is rational over $\mathbb{C}$.

The degree of $\pi_1 : Y \to X$ can be computed as follows: It is the number of secants to Σ through a generic point $p \in X$. There is one such secant by Lemma 3. We will return to this in Sect. 4.

Consider the nested Hilbert scheme

$$\mathrm{Scr} = \{\Sigma \subset X \subset \mathbb{P}^5 : \Sigma \text{ quartic scroll, } X \text{ cubic fourfold}\}$$

and let $\pi : \mathrm{Scr} \to \mathbb{P}^{55}$ denote the morphism forgetting Σ. We have $\dim(\mathrm{Scr}) = 56$ so the fibers of π are positive dimensional. In 1940, Morin [Mor40] asserted that the generic fiber of π is one dimensional, deducing (incorrectly!) that the generic cubic fourfold is rational. Fano [Fan43] corrected this a few years later, showing that the generic fiber has dimension two; cubic fourfolds containing a quartic scroll thus form a *divisor* in $\mathcal{C}$. We will develop a conceptual approach to this in Sect. 2.1.

1.5 Cubic Fourfolds Containing a Quintic del Pezzo Surface

Let $T \subset \mathbb{P}^5$ denote a quintic del Pezzo surface, i.e., $T = \mathrm{Bl}_{p_1,p_2,p_3,p_4}(\mathbb{P}^2)$ anti-canonically embedded. Its defining equations are quadrics

$$Q_i = a_{jk}a_{lm} - a_{jl}a_{km} + a_{jm}a_{kl}, \{1,\dots,5\} = \{i,j,k,l,m\}, j < k < l < m,$$

where the a_{rs} are generic linear forms on $\mathbb{P}^5$. The rational map

$$\begin{aligned} Q : \mathbb{P}^5 &\dashrightarrow \mathbb{P}^4 \\ [u,v,w,x,y,z] &\mapsto [Q_1,Q_2,Q_3,Q_4,Q_5] \end{aligned}$$

contracts each secant of T to a point. Note that a generic $p \in \mathbb{P}^5$ lies on a unique such secant.

Proposition 6 *A cubic fourfold containing a quintic del Pezzo surface is rational.*

Indeed, restricting Q to X yields a *birational* morphism $\mathrm{Bl}_T(X) \to \mathbb{P}^4$.

1.6 Pfaffian Cubic Fourfolds

Recall that if $M = (m_{ij})$ is skew-symmetric $2n \times 2n$ matrix then the determinant

$$\det(M) = \mathrm{Pf}(M)^2,$$

where $\mathrm{Pf}(M)$ is a homogeneous form of degree n in the entries of M, known as its Pfaffian. If the entries of M are taken as linear forms in u,v,w,x,y,z, the resulting hypersurface

$$X = \{\mathrm{Pf}(M) = 0\} \subset \mathbb{P}^5$$

is a Pfaffian cubic fourfold.

We put this on a more systematic footing. Let V denote a six-dimensional vector space and consider the strata

$$\mathrm{Gr}(2, V) \subset \mathrm{Pfaff}(V) \subset \mathbb{P}(\bigwedge^2 V),$$

where $\mathrm{Pfaff}(V)$ parametrizes the rank-four tensors. Note that $\mathrm{Pfaff}(V)$ coincides with the secant variety to $\mathrm{Gr}(2, V)$, which is degenerate, i.e., smaller than the expected dimension. We also have dual picture

$$\mathrm{Gr}(2, V^*) \subset \mathrm{Pfaff}(V^*) \subset \mathbb{P}(\bigwedge^2 V^*).$$

A codimension six subspace $L \subset \mathbb{P}(\bigwedge^2 V)$ corresponds to a codimension nine subspace $L^\perp \subset \mathbb{P}(\bigwedge^2 V^*)$. Let $X = L^\perp \cap \mathrm{Pfaff}(V^*)$ denote the resulting Pfaffian cubic fourfold and $S = L \cap \mathrm{Gr}(2, V)$ the associated degree fourteen K3 surface.

Beauville and Donagi [BD85] (see also [Tre84]) established the following properties, when L is generically chosen:

- X is rational: For each codimension one subspace $W \subset V$, the mapping

$$\begin{aligned} Q_W : X &\dashrightarrow W \\ [\phi] &\mapsto \ker(\phi) \cap W \end{aligned}$$

  is birational. Here we interpret $\phi : V \to V^*$ as an antisymmetric linear transformation.
- X contains quartic scrolls: For each point $[P] \in S$, consider

$$\Sigma_P := \{[\phi] \in X : \ker(\phi) \cap P \neq 0\}.$$

  We interpret $P \subset V$ as a linear subspace. This is the two-parameter family described by Fano.
- X contains quintic del Pezzo surfaces: For each W, consider

$$T_W := \{[\phi] \in X : \ker(\phi) \subset W\},$$

  the indeterminacy of Q_W. This is a five-parameter family.
- The variety $F_1(X)$ of lines on X is isomorphic to $S^{[2]}$, the Hilbert scheme of length two subschemes on S.

Tregub [Tre93] observed the connection between containing a quartic scroll and containing a quintic del Pezzo surface. For the equivalence between containing a quintic del Pezzo surface and the Pfaffian condition, see [Bea00, Proposition 9.2(a)].

Remark 7 Cubic fourfolds X containing disjoint planes P_1 and P_2 admit 'degenerate' quartic scrolls and are therefore limits of Pfaffian cubic fourfolds [Tre93]. As we saw in Sect. 1.2, the lines connecting P_1 and P_2 and contained in X are parametrized by a K3 surface

$$S \subset P_1 \times P_2.$$

Given $s \in S$ generic, let ℓ_s denote the corresponding line and $L_i = \mathrm{span}(P_i, \ell_s) \simeq \mathbb{P}^3$. The intersection

$$L_i \cap X = P_i \cup Q_i$$

where Q_i is a quadric surface. The surfaces Q_1 and Q_2 meet along the common ruling ℓ_s, hence $Q_1 \cup_{\ell_s} Q_2$ is a limit of quartic scrolls.

Remark 8 (Limits of Pfaffians) A number of recent papers have explored smooth limits of Pfaffian cubic fourfolds more systematically. For analysis of the intersection between cubic fourfolds containing a plane and limits of the Pfaffian locus, see [ABBVA14]. Auel and Bolognese-Russo-Staglianò [BRS15] have shown that smooth limits of Pfaffian cubic fourfolds are always rational; [BRS15] includes a careful analysis of the topology of the Pfaffian locus in moduli.

1.7 General Geometric Properties of Cubic Hypersurfaces

Let $\mathrm{Gr}(2, n+1)$ denote the Grassmannian of lines in $\mathbb{P}^n$. We have a tautological exact sequence

$$0 \to S \to \mathcal{O}^{n+1}_{\mathrm{Gr}(2,n+1)} \to Q \to 0$$

where S and Q are the tautological sub- and quotient bundles, of ranks 2 and $n-1$. For a hypersurface $X \subset \mathbb{P}^n$, the variety of lines $F_1(X) \subset \mathrm{Gr}(2, n+1)$ parametrizes lines contained in X. If $X = \{G = 0\}$ for some homogeneous form G of degree $d = \deg(X)$ then $F_1(X) = \{\sigma_G = 0\}$, where

$$\sigma_G \in \Gamma(\mathrm{Gr}(2, n+1), \mathrm{Sym}^d(S^*))$$

is the image of G under the dth symmetric power of the transpose to $S \hookrightarrow \mathcal{O}^{n+1}_{\mathrm{Gr}(2,n+1)}$.

Proposition 9 ([AK77, Theorem 1.10]) *Let $X \subset \mathbb{P}^n, n \geq 3$, be a smooth cubic hypersurface. Then $F_1(X)$ is smooth of dimension $2n-6$.*

The proof is a local computation on tangent spaces.

Proposition 10 *Let $\ell \subset X \subset \mathbb{P}^n$ be a smooth cubic hypersurface containing a line. Then X admits a degree two unirational parametrization, i.e., a degree two mapping*

$$\rho : \mathbb{P}^{n-1} \dashrightarrow X.$$

Since this result is classical we only sketch the key ideas. Consider the diagram

$$\begin{tikzcd} & \mathrm{Bl}_\ell(\mathbb{P}^n) \arrow[dl] \arrow[dr] & \\ \mathbb{P}^n \arrow[rr, dashed] & & \mathbb{P}^{n-2} \end{tikzcd}$$

where the bottom arrow is projection from ℓ. The right arrow is a $\mathbb{P}^2$ bundle. This induces

$$\begin{tikzcd} & \mathrm{Bl}_\ell(X) \arrow[dl] \arrow[dr, "q"] & \\ X \arrow[rr, dashed] & & \mathbb{P}^{n-2} \end{tikzcd}$$

where q is a conic bundle. The exceptional divisor $E \simeq \mathbb{P}(N_{\ell/X}) \subset \mathrm{Bl}_\ell(X)$ meets each conic fiber in two points. Thus the base change

$$Y := \mathrm{Bl}_\ell(X) \times_{\mathbb{P}^{n-2}} E \to E$$

has a rational section and we obtain birational equivalences

$$Y \overset{\sim}{\dashrightarrow} \mathbb{P}^1 \times E \overset{\sim}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^{n-3} \times \ell \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-1}.$$

The induced $\rho : Y \dashrightarrow X$ is generically finite of degree two.

2 Special Cubic Fourfolds

We use the terminology *very general* to mean ‘outside the union of a countable collection of Zariski-closed subvarieties’. Throughout this section, X denotes a smooth cubic fourfold over $\mathbb{C}$.

2.1 Structure of Cohomology

Let X be a cubic fourfold and $h \in H^2(X, \mathbb{Z})$ the Poincaré dual to the hyperplane class, so that $h^4 = \deg(X) = 3$. The Lefschetz hyperplane theorem and Poincaré duality give

$$H^2(X, \mathbb{Z}) = \mathbb{Z}h, \quad H^6(X, \mathbb{Z}) = \mathbb{Z}\frac{h^3}{3}.$$

The Hodge numbers of X take the form

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array}$$

so the Hodge-Riemann bilinear relations imply that $H^4(X, \mathbb{Z})$ is a unimodular lattice under the intersection form $\langle , \rangle$ of signature $(21, 2)$. Basic classification results on quadratic forms [Has00, Proposition 2.1.2] imply

$$L := H^4(X, \mathbb{Z})_{\langle , \rangle} \simeq (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}.$$

The primitive cohomology

$$L^0 := \{h^2\}^{\perp} \simeq A_2 \oplus U^{\oplus 2} \oplus E_8^{\oplus 2},$$

where

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E_8 is the positive definite lattice associated with the corresponding Dynkin diagram. This can be shown using the incidence correspondence between X and its variety of lines $F_1(X)$, which induces the *Abel-Jacobi* mapping [BD85]

$$\alpha : H^4(X, \mathbb{Z}) \to H^2(F_1(X), \mathbb{Z})(-1), \tag{1}$$

compatible with Hodge filtrations. (See [AT14, Sect. 2] for another approach.) Restricting to primitive cohomology gives an isomorphism

$$\alpha : H^4(X, \mathbb{Z})_{\text{prim}} \xrightarrow{\sim} H^2(F_1(X), \mathbb{Z})_{\text{prim}}(-1).$$

Note that $H^2(F_1(X), \mathbb{Z})$ carries the *Beauville-Bogomolov* form $(,)$; see Sect. 3.5 for more discussion. The shift in weights explains the change in signs

$$(\alpha(x_1), \alpha(x_2)) = -\langle x_1, x_2 \rangle .$$

2.2 Special Cubic Fourfolds Defined

For a very general cubic fourfold X, any algebraic surface $S \subset X$ is homologous to a complete intersection, i.e.,

$$H^{2,2}(X, \mathbb{Z}) := H^4(X, \mathbb{Z}) \cap H^2(\Omega_X^2) \simeq \mathbb{Z}h^2$$

so

$$[S] \equiv nh^2, \quad n = \deg(S)/3.$$

This follows from the Torelli Theorem and the irreducibility of the monodromy action for cubic fourfolds [Voi86]; see Sect. 2.3 below for more details. In particular, X does not contain any quartic scrolls or any surfaces of degree four; this explains why Morin's rationality argument could not be correct.

On the other hand, the integral Hodge conjecture is valid for cubic fourfolds [Voi13, Theorem 1.4], so every class

$$\gamma \in H^{2,2}(X, \mathbb{Z})$$

is algebraic, i.e., arises from a codimension two algebraic cycle with integral coefficients. Thus if

$$H^{2,2}(X, \mathbb{Z}) \supsetneq \mathbb{Z}h^2$$

then X admits surfaces that are not homologous to complete intersections.

Definition 11 A cubic fourfold X is *special* if it admits an algebraic surface $S \subset X$ not homologous to a complete intersection. A *labelling* of a special cubic fourfold consists of a rank two saturated sublattice

$$h^2 \in K \subset H^{2,2}(X, \mathbb{Z});$$

its *discriminant* is the determinant of the intersection form on K.

Let $S \subset X$ be a smooth surface. Recall that

$$c_1(T_X) = 3h, \quad c_2(T_X) = 6h^2$$

so the self-intersection

$$\begin{aligned}\langle S,S\rangle &= c_2(N_{S/X}) = c_2(T_X|S) - c_2(T_S) - c_1(T_S)c_1(T_X|S) + c_1(T_S)^2\\ &= 6H^2 + 3HK_S + K_S^2 - \chi(S),\end{aligned}$$

where $H = h|S$ and $\chi(S)$ is the topological Euler characteristic.

(1) When X contains a plane P we have

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}.$$

(2) When X contains a cubic scroll Σ_3, i.e., $\mathrm{Bl}_p(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$, we have

$$K_{12} = \begin{array}{c|cc} & h^2 & \Sigma_3 \\ \hline h^2 & 3 & 3 \\ \Sigma_3 & 3 & 7 \end{array}.$$

(3) When X contains a quartic scroll Σ_4 or a quintic del Pezzo surface T then we have

$$K_{14} = \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array} \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 5 \\ T & 5 & 13 \end{array}, \quad T = 3h^2 - \Sigma_4.$$

We return to cubic fourfolds containing two disjoint planes P_1 and P_2. Here we have a rank three lattice of algebraic classes, containing a distinguished rank two lattice:

$$\begin{array}{c|ccc} & h^2 & P_1 & P_2 \\ \hline h^2 & 3 & 1 & 1 \\ P_1 & 1 & 3 & 0 \\ P_2 & 1 & 0 & 3 \end{array} \supset \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array}, \quad \Sigma_4 = 2h^2 - P_1 - P_2.$$

2.3 Structural Results

Voisin's Torelli Theorem and the geometric description of the period domains yields a qualitative description of the special cubic fourfolds.

Consider cubic fourfolds X with a prescribed saturated sublattice

$$h^2 \in M \subset L \simeq H^4(X,\mathbb{Z})$$

of algebraic classes. The Hodge-Riemann bilinear relations imply that M is positive definite. Then the Hodge structure on X is determined by $H^1(X, \Omega_X^3) \subset M^\perp \otimes \mathbb{C}$, which is isotropic for $\langle , \rangle$. The relevant period domain is

$$\mathcal{D}_M = \{[\lambda] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) : \langle \lambda, \lambda \rangle = 0\},$$

or at least the connected component with the correct orientation. (See [Mar11, Sect. 4] for more discussion of orientations.) The Torelli theorem [Voi86] asserts that the period map

$$\begin{aligned} \tau : \mathcal{C} &\to \Gamma \backslash \mathcal{D}_{\mathbb{Z}h^2} \\ X &\mapsto H^1(X, \Omega_X^3) \end{aligned}$$

is an open immersion; Γ is the group of automorphisms of the primitive cohomology lattice L^0 arising from the monodromy of cubic fourfolds. Cubic fourfolds with additional algebraic cycles, indexed by a saturated sublattice

$$M \subsetneq M' \subset L,$$

correspond to the linear sections of $\mathcal{D}_M$ of codimension $\operatorname{rank}(M'/M)$.

Proposition 12 *Fix a positive definite lattice M of rank r admitting a saturated embedding*

$$h^2 \in M \subset L.$$

If this exists then $M^0 = \{h^2\}^\perp \subset M$ is necessarily even, as it embeds in L^0.

Let $\mathcal{C}_M \subset \mathcal{C}$ denote the cubic fourfolds X admitting algebraic classes with this lattice structure

$$h^2 \in M \subset H^{2,2}(X, \mathbb{Z}) \subset L.$$

Then $\mathcal{C}_M$ has codimension $r - 1$, provided it is non-empty.

We can make this considerably more precise in rank two. For each labelling K, pick a generator $K \cap L^0 = \mathbb{Z}v$. Classifying orbits of primitive positive vectors $v \in L^0$ under the automorphisms of this lattice associated with the monodromy representation yields:

Theorem 13 ([Has00, Sect. 3]) *Let $\mathcal{C}_d \subset \mathcal{C}$ denote the special cubic fourfolds admitting a labelling of discriminant d. Then $\mathcal{C}_d$ is non-empty if and only if $d \geq 8$ and $d \equiv 0, 2 \pmod 6$. Moreover, $\mathcal{C}_d$ is an irreducible divisor in $\mathcal{C}$.*

Fix a discriminant d and consider the locus $\mathcal{C}_d \subset \mathcal{C}$. The Torelli Theorem implies that irreducible components of $\mathcal{C}_d$ correspond to saturated rank two sublattices realizations

$$h^2 \in K \subset L$$

up to monodromy. The monodromy of cubic fourfolds acts on L via $\mathrm{Aut}(L, h^2)$, the automorphisms of the lattice L preserving h^2. Standard results on embeddings of lattices imply there is a unique $K \subset L$ modulo $\mathrm{Aut}(L, h^2)$. The monodromy group is an explicit finite index subgroup of $\mathrm{Aut}(L, h^2)$, which still acts transitively on these sublattices. Hence $\mathcal{C}_d$ is irreducible.

The rank two lattices associated with labellings of discriminant d are:

$$K_d := \begin{cases} \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 1 \\ S & 1 & 2n+1 \end{array} & \text{if } d = 6n+2, n \geq 1 \\ \\ \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 0 \\ S & 0 & 2n \end{array} & \text{if } d = 6n, n \geq 2. \end{cases}$$

The cases $d = 2$ and 6

$$K_2 = \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 1 \\ S & 1 & 1 \end{array} \quad K_6 = \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 0 \\ S & 0 & 2 \end{array} \tag{2}$$

correspond to limiting Hodge structures arising from singular cubic fourfolds: the symmetric determinant cubic fourfolds [Has00, Sect. 4.4] and the cubic fourfolds with an ordinary double point [Has00, Sect. 4.2]. The non-special cohomology lattice $K_d^{\perp}$ is also well-defined for all $(X, K_d) \in \mathcal{C}_d$.

Laza [Laz10], building on work of Looijenga [Loo09], gives precise necessary and sufficient conditions for when the $\mathcal{C}_M$ in Proposition 12 are nonempty:

- M is positive definite and admits a saturated embedding $h^2 \in M \subset L$;
- there exists no sublattice $h^2 \in K \subset M$ with $K \simeq K_2$ or K_6 as in (2).

Detailed descriptions of the possible lattices of algebraic classes are given by Mayanskiy [May11]. Furthermore, Laza obtains a characterization of the image of the period map for cubic fourfolds: it is complement of the divisors parametrizing ‘special’ Hodge structures with a labelling of discriminant 2 or 6.

Remark 14 Li and Zhang [LZ13] have found a beautiful generating series for the degrees of special cubic fourfolds of discriminant d, expressed via modular forms.

We have seen concrete descriptions of surfaces arising in special cubic fourfolds for $d = 8, 12, 14$. Nuer [Nue15, Sect. 3], [Nue17] writes down explicit smooth rational surfaces arising in generic special cubic fourfolds of discriminants $d \leq 38$. These are blow-ups of the plane at points in general position, embedded via explicit linear series, e.g.,

1. For $d = 18$, let S be a generic projection into $\mathbb{P}^5$ of a sextic del Pezzo surface in $\mathbb{P}^6$.
2. For $d = 20$, let S be a Veronese embedding of $\mathbb{P}^2$.

Question 15 Is the algebraic cohomology of a special cubic fourfold generated by the classes of smooth rational surfaces?

Voisin has shown that the cohomology can be generated by smooth surfaces (see the proof of [Voi14, Theorem 5.6]) *or* by possibly singular rational surfaces [Voi07]. Low discriminant examples suggest we might be able to achieve both.

A by-product of Nuer's approach, where it applies, is to prove that the $\mathcal{C}_d$ are unirational. However, for $d \gg 0$ the loci $\mathcal{C}_d$ are of general type [TVA15]. So a different approach is needed in general.

2.4 Census of Rational Cubic Fourfolds

Using this framework, we enumerate the smooth cubic fourfolds known to be rational:

1. cubic fourfolds in $\mathcal{C}_{14}$, the closure of the Pfaffian locus;
2. cubic fourfolds in $\mathcal{C}_8$, the locus containing a plane P, such that there exists a class W such that $\langle W, (h^2 - P)\rangle$ is odd.

For the second case, note that the discriminant of the lattice $M = \mathbb{Z}h^2 + \mathbb{Z}P + \mathbb{Z}W$ has the same parity as $\langle W, (h^2 - P)\rangle$.

Thus all the cubic fourfolds proven to be rational are parametrized by one divisor $\mathcal{C}_{14}$ and a countably-infinite union of codimension two subvarieties $\mathcal{C}_M \subset \mathcal{C}_8$.

Question 16 Is there a rational (smooth) cubic fourfold not in the enumeration above?

New rational cubic fourfolds were found by Addington, Tschinkel, Várilly-Alavardo, and the author in 'Cubic fourfolds fibered in sextic del Pezzo surfaces' arXiv:1606.05321.

There are conjectural frameworks (see Sect. 3.3 and also Sect. 3.6) predicting that many special cubic fourfolds should be rational. However, few new examples of cubic fourfolds have been found to support these frameworks.

3 Associated K3 Surfaces

3.1 Motivation

The motivation for considering associated K3 surfaces comes from the Clemens-Griffiths [CG72] proof of the irrationality of cubic threefolds. Suppose X is a rational threefold. Then we have an isomorphism of polarized Hodge structures

$$H^3(X,\mathbb{Z}) = \oplus_{i=1}^n H^1(C_i,\mathbb{Z})(-1)$$

where the C_i are smooth projective curves. Essentially, the C_i are blown up in the birational map

$$\mathbb{P}^3 \dashrightarrow X.$$

If X is a rational *fourfold* then we can look for the cohomology of surfaces blown up in the birational map

$$\rho : \mathbb{P}^4 \dashrightarrow X.$$

Precisely, if P is a smooth projective fourfold, $S \subset P$ an embedded surface, and $\tilde{P} = \mathrm{Bl}_S(P)$ then we have [Ful84, Sect. 6.7]

$$H^4(\tilde{P},\mathbb{Z}) \simeq H^4(P,\mathbb{Z}) \oplus_\perp H^2(S,\mathbb{Z})(-1).$$

The homomorphism $H^4(P,\mathbb{Z}) \to H^4(\tilde{P},\mathbb{Z})$ is induced by pull-back; the homomorphism $H^2(S,\mathbb{Z})(-1) \to H^4(\tilde{P},\mathbb{Z})$ comes from the composition of pull-back and push-forward

$$\begin{tikzcd} E = \mathbb{P}(N_{S/P}) \arrow[r, hook] \arrow[d] & \tilde{P} \\ S & \end{tikzcd}$$

where $N_{S/P}$ is the normal bundle and E the exceptional divisor. Blowing up points in P contributes Hodge-Tate summands $\mathbb{Z}(-2)$ with negative self-intersection to its middle cohomology; these have the same affect as blowing up a surface (like $\mathbb{P}^2$) with $H^2(S,\mathbb{Z}) \simeq \mathbb{Z}$. Blowing up curves does not affect middle cohomology.

Applying the Weak Factorization Theorem [Wło03, AKMW02]—that every birational map is a composition of blow-ups and blow-downs along smooth centers—we obtain the following:

Proposition 17 *Suppose X is a rational fourfold. Then there exist smooth projective surfaces $S_1, \ldots, S_n$ and $T_1, \ldots, T_m$ such that we have an isomorphism of Hodge structures*

$$H^4(X, \mathbb{Z}) \oplus (\oplus_{j=1,\ldots,m} H^2(T_j, \mathbb{Z})(-1)) \simeq \oplus_{i=1,\ldots,n} H^2(S_i, \mathbb{Z})(-1).$$

Unfortunately, it is not clear how to extract a computable invariant from this observation; but see [ABGvB13, Kul08] for work in this direction.

3.2 Definitions and Reduction to Lattice Equivalences

In light of examples illustrating how rational cubic fourfolds tend to be entangled with K3 surfaces, it is natural to explore this connection on the level of Hodge structures.

Let (X, K) denote a labelled special cubic fourfold. A polarized K3 surface (S, f) is *associated* with (X, K) if there exists an isomorphism of lattices

$$H^4(X, \mathbb{Z}) \supset K^{\perp} \xrightarrow{\sim} f^{\perp} \subset H^2(S, \mathbb{Z})(-1)$$

respecting Hodge structures.

Example 18 (Pfaffians) Let X be a Pfaffian cubic fourfold (see Sect. 1.6) and

$$K_{14} = \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array} \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 5 \\ T & 5 & 13 \end{array},$$

the lattice containing the classes of the resulting quartic scrolls and quintic del Pezzo surfaces. Let (S, f) be the K3 surface of degree 14 arising from the Pfaffian construction. Then (S, f) is associated with (X, K_{14}).

Example 19 (A Suggestive Non-example) As we saw in Sect. 1.6, a cubic fourfold X containing a plane P gives rise to a degree two K3 surface (S, f). However, this is *not* generally associated with the cubic fourfold. If $K_8 \subset H^4(X, \mathbb{Z})$ is the labelling then

$$K_8^{\perp} \subset f^{\perp}$$

as an index two sublattice [vG05, Sect. 9.7]. However, when the quadric bundle $q : \mathrm{Bl}_P(X) \to \mathbb{P}^2$ admits a section (so that X is rational), S often admits a polarization g such that (S, g) is associated with some labelling of X. See [Kuz10, Kuz17] for further discussion.

Proposition 20 *The existence of an associated K3 surface depends only on the discriminant of the rank two lattice K.*

Here is an outline of the proof; we refer to [Has00, Sect. 5] for details.

Recall the discussion of Theorem 13 in Sect. 2.3: For each discriminant $d \equiv 0, 2 \pmod 6$ with $d > 6$, there exists a lattice $K_d^\perp$ such that each special cubic fourfold of discriminant d (X, K) has $K^\perp \simeq K_d^\perp$. Consider the primitive cohomology lattice

$$\Lambda_d := f^\perp \subset H^2(S, \mathbb{Z})(-1)$$

for a polarized K3 surface (S, f) of degree d. The moduli space $\mathcal{N}_d$ of such surfaces is connected, so Λ_d is well-defined up to isomorphism.

We claim (X, K_d) admits an associated K3 surface if and only if there exists an isomorphism of lattices

$$\iota : K_d^\perp \xrightarrow{\sim} \Lambda_d.$$

This is clearly a necessary condition. For sufficiency, given a Hodge structure on Λ_d surjectivity of the Torelli map for K3 surfaces [Siu81] ensures there exists a K3 surface S and a divisor f with that Hodge structure. It remains to show that f can be taken to be a polarization of S, i.e., there are no (-2)-curves orthogonal to f. After twisting and applying ι, any such curve yields an algebraic class $R \in H^{2,2}(X, \mathbb{Z})$ with $\langle R, R \rangle = 2$ and $\langle h^2, R \rangle = 0$. In other words, we obtain a labelling

$$K_6 = \langle h^2, R \rangle \subset H^4(X, \mathbb{Z}).$$

Such labellings are associated with nodal cubic fourfolds, violating the smoothness of X.

Based on this discussion, it only remains to characterize when the lattice isomorphism exists. Nikulin's theory of discriminant forms [Nik79] yields:

Proposition 21 ([Has00, Proposition 5.1.4]) *Let d be a positive integer congruent to* 0 *or* 2 *modulo* 6*. Then there exists an isomorphism*

$$\iota : K_d^\perp \xrightarrow{\sim} \Lambda_d(-1)$$

if and only if d is not divisible by $4, 9$ *or any odd prime congruent to* 2 *modulo* 3*.*

Definition 22 An even integer $d > 0$ is *admissible* if it is not divisible by $4, 9$ or any odd prime congruent to 2 modulo 3.

Thus we obtain:

Theorem 14 *A special cubic fourfold (X, K_d) admits an associated K3 surface if and only if d is admissible.*

3.3 Connections with Kuznetsov's Philosophy

Kuznetsov has proposed a criterion for rationality expressed via derived categories [Kuz10, Conjecture 1.1] [Kuz17]: Let X be a cubic fourfold, $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X, and $\mathcal{A}_X$ the subcategory orthogonal to the exceptional collection $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$. Kuznetsov proposes that X is rational if and only if $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface. He verifies this for the known examples of rational cubic fourfolds.

Addington and Thomas [AT14, Theorem 1.1] show that the generic $(X, K_d) \in \mathcal{C}_d$ satisfies Kuznetsov's derived category condition precisely when d is admissible. In Sect. 3.7 we present some of the geometry behind this result. Thus we find:

> Kuznetsov's conjecture would imply that the generic $(X, K_d) \in \mathcal{C}_d$ for admissible d is rational.

In particular, special cubic fourfolds of discriminants $d = 26, 38, 42, \ldots$ would all be rational!

3.4 Naturality of Associated K3 Surfaces

There are a priori many choices for the lattice isomorphism ι so a given cubic fourfold could admit several associated K3 surfaces. Here we will analyze this more precisely. Throughout, assume that d is admissible.

We require a couple variations on $\mathcal{C}_d \subset \mathcal{C}$:

- Let $\mathcal{C}_d^\nu$ denote labelled cubic fourfolds, with a saturated lattice $K \ni h^2$ of algebraic classes of rank two and discriminant d.
- Let $\mathcal{C}'_d$ denote pairs consisting of a cubic fourfold X and a saturated embedding of K_d into the algebraic cohomology.

We have natural maps

$$\mathcal{C}'_d \to \mathcal{C}_d^\nu \to \mathcal{C}_d.$$

The second arrow is normalization over cubic fourfolds admitting multiple labellings of discriminant d. To analyze the first arrow, note that the K_d admits non-trivial automorphisms fixing h^2 if and only if $6|d$. Thus the first arrow is necessarily an isomorphism unless $6|d$. When $6|d$ the lattice K_d admits an automorphism acting by multiplication by -1 on the orthogonal complement of h^2. $\mathcal{C}'_d$ is irreducible if this involution can be realized in the monodromy group. An analysis of the monodromy group gives:

Proposition 24 ([Has00, Sect. 5]) *For each admissible $d > 6$, $\mathcal{C}'_d$ is irreducible and admits an open immersion into the moduli space $\mathcal{N}_d$ of polarized K3 surfaces of degree d.*

Corollary 25 *Assume $d > 6$ is admissible. If $d \equiv 2 \pmod 6$ then $\mathcal{C}_d$ is birational to $\mathcal{N}_d$. Otherwise $\mathcal{C}_d$ is birational to a quotient of $\mathcal{N}_d$ by an involution.*

Thus for $d = 42, 78, \ldots$ cubic fourfolds $X \in \mathcal{C}_d$ admit *two* associated K3 surfaces.

Even the open immersions from the double covers

$$j_{\iota,d} : \mathcal{C}'_d \hookrightarrow \mathcal{N}_d$$

are typically not canonical. The possible choices correspond to orbits of the isomorphism

$$\iota : K_d^{\perp} \xrightarrow{\sim} \Lambda_d(-1)$$

under postcomposition by automorphisms of Λ_d coming from the monodromy of K3 surfaces and precomposition by automorphisms of $K_d^{\perp}$ coming from the subgroup of the monodromy group of cubic fourfolds fixing the elements of K_d.

Proposition 26 ([Has00, Sect. 5.2]) *Choices of*

$$j_{\iota,d} : \mathcal{C}'_d \hookrightarrow \mathcal{N}_d$$

are in bijection with the set

$$\{a \in \mathbb{Z}/d\mathbb{Z} : a^2 \equiv 1 \pmod{2d}\}/\pm 1.$$

If d is divisible by r distinct odd primes then there are 2^{r-1} possibilities.

Remark 27 The ambiguity in associated K3 surfaces can be expressed in the language of equivalence of derived categories. Suppose that (S_1, f_1) and (S_2, f_2) are polarized K3 surfaces of degree d, both associated with a special cubic fourfold of discriminant d. This means we have an isomorphism of Hodge structures

$$H^2(S_1, \mathbb{Z})_{\text{prim}} \simeq H^2(S_2, \mathbb{Z})_{\text{prim}}$$

so their transcendental cohomologies are isomorphic. Orlov's Theorem [Orl97, Sect. 3] implies S_1 and S_2 are *derived equivalent*, i.e., their bounded derived categories of coherent sheaves are equivalent.

Proposition 26 may be compared with the formula counting derived equivalent K3 surfaces in [HLOY03]. We will revisit this issue in Sect. 3.7.

3.5 Interpreting Associated K3 Surfaces I

We offer geometric interpretations of associated K3 surfaces. These are most naturally expressed in terms of moduli spaces of sheaves on K3 surfaces.

Let M be an irreducible holomorphic symplectic variety, i.e., a smooth simply connected projective variety such that $\Gamma(M, \Omega^2_M) = \mathbb{C}\omega$ where ω is everywhere nondegenerate. The cohomology $H^2(M, \mathbb{Z})$ admits a distinguished integral quadratic form $(,)$, called the *Beauville-Bogomolov* form [Bea83]. Examples include:

- K3 surfaces S with $(,)$ the intersection form;
- Hilbert schemes $S^{[n]}$ of length n zero dimensional subschemes on a K3 surface S, with

$$H^2(S^{[n]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta, \quad (\delta, \delta) = -2(n-1), \tag{3}$$

where 2δ parametrizes the non-reduced subschemes.

Example 28 Let (S, f) be a generic degree 14 K3 surface. Then $S^{[2]} \simeq F_1(X) \subset \mathrm{Gr}(2, 6)$ where X is a Pfaffian cubic fourfold (see Sect. 1.6). The polarization induced from the Grassmannian is $2f - 5\delta$; note that

$$(2f - 5\delta, 2f - 5\delta) = 4 \cdot 14 - 25 \cdot 2 = 6.$$

The example implies that if $F_1(X) \subset \mathrm{Gr}(2, 6)$ is the variety of lines on an arbitrary cubic fourfold then the polarization $g = \alpha(h^2)$ satisfies

$$(g, g) = 6, \quad \big(g, H^2(F_1(X), \mathbb{Z})\big) = 2\mathbb{Z}.$$

It follows that the Abel-Jacobi map is an isomorphism of *abelian groups*

$$\alpha : H^4(X, \mathbb{Z}) \to H^2(F_1(X), \mathbb{Z})(-1) \tag{4}$$

Indeed, α is an isomorphism on primitive cohomology and both

$$\mathbb{Z}h^2 \oplus H^4(X, \mathbb{Z})_{prim} \subset H^4(X, \mathbb{Z})$$

and

$$\mathbb{Z}g \oplus H^2(F_1(X), \mathbb{Z})_{prim} \subset H^2(F(X), \mathbb{Z})$$

have index three as subgroups.

The Pfaffian case is the first of an infinite series of examples:

Theorem 29 ([Has00, Sect. 6] [Add16, Theorem 2]) *Fix an integer of the form $d = 2(n^2+n+1)/a^2$, where $n > 1$ and $a > 0$ are integers. Let X be a cubic fourfold* *in $\mathcal{C}_d$ with variety of lines $F_1(X)$. Then there exists a polarized K3 surface (S,f) of degree d and a birational map*

$$F_1(X) \overset{\sim}{\dashrightarrow} S^{[2]}.$$

If $a = 1$ and $X \in \mathcal{C}_d$ in generic then $F_1(X) \simeq S^{[2]}$ with polarization $g = 2f - (2n + 1)\delta$.

The first part relies on Verbitsky's global Torelli theorem for hyperkähler manifolds [Mar11]. The last assertion is proven via a degeneration/specialization argument along the nodal cubic fourfolds, which correspond to degree six K3 surfaces (S', f'). We specialize so that $F = S'^{[2]}$ admits involutions not arising from involutions of S'. Thus the deformation space of F admits several divisors parametrizing Hilbert schemes of K3 surfaces.

Since the primitive cohomology of S sits in $H^2(S^{[2]}, \mathbb{Z})$, the Abel-Jacobi map (1) explains why S is associated with X. If $3|d$ ($d \neq 6$) then Theorem 29 and Corollary 25 yield two K3 surfaces S_1 and S_2 such that

$$F_1(X) \simeq S_1^{[2]} \simeq S_2^{[2]}.$$

With a view toward extending this argument, we compute the cohomology of the varieties of lines of special cubic fourfolds. This follows immediately from (4):

Proposition 30 *Let (X, K_d) be a special cubic fourfold of discriminant d, $F_1(X) \subset \mathrm{Gr}(2,6)$ its variety of lines, and $g = \alpha(h^2)$ the resulting polarization. Then $\alpha(K_d)$ is saturated in $H^2(F_1(X), \mathbb{Z})$ and*

$$\alpha(K_d) \simeq \begin{cases} \begin{array}{c|cc} & g & T \\ \hline g & 6 & 0 \\ T & 0 & -2n \end{array} & \text{if } d = 6n \\ \begin{array}{c|cc} & g & T \\ \hline g & 6 & 2 \\ T & 2 & -2n \end{array} & \text{if } d = 6n + 2. \end{cases}$$

The following example shows that Hilbert schemes are insufficient to explain all associated K3 surfaces:

Example 31 Let $(X, K_{74}) \in \mathcal{C}_{74}$ be a generic point, which admits an associated K3 surface by Theorem 14. There does not exist a K3 surface S with $F_1(X) \simeq S^{[2]}$, even birationally. Indeed, the $H^2(M, \mathbb{Z})$ is a birational invariant of holomorphic symplectic manifolds but

$$\alpha(K_{74}) \simeq \begin{pmatrix} 6 & 2 \\ 2 & -24 \end{pmatrix}$$

is not isomorphic to the Picard lattice of the Hilbert scheme

$$\mathrm{Pic}(S^{[2]}) \simeq \begin{pmatrix} 74 & 0 \\ 0 & -2 \end{pmatrix}.$$

Addington [Add16] gives a systematic discussion of this issue.

3.6 Derived Equivalence and Cremona Transformations

In light of the ambiguity of associated K3 surfaces (see Remark 27) and the general discussion in Sect. 3.1, it is natural to seek diagrams

$$\begin{tikzcd} & \mathrm{Bl}_{S_1}(\mathbb{P}^4) \arrow[r, "\sim"] \arrow[dl, "\beta_1"'] & \mathrm{Bl}_{S_2}(\mathbb{P}^4) \arrow[dr, "\beta_2"] & \\ \mathbb{P}^4 & & & \mathbb{P}^4 \end{tikzcd} \tag{5}$$

where β_i is the blow-up along a smooth surface S_i, with S_1 and S_2 derived equivalent but not birational.

Cremona transformations of $\mathbb{P}^4$ with smooth surfaces as their centers have been classified by Crauder and Katz [CK89, Sect. 3]; possible centers are either quintic elliptic scrolls or surfaces $S \subset \mathbb{P}^4$ of degree ten given by the vanishing of the 4×4 minors of a 4×5 matrix of linear forms. A generic surface of the latter type admits divisors

$$\begin{array}{c|cc} & K_S & H \\ \hline K_S & 5 & 10 \\ H & 10 & 10 \end{array},$$

where H is the restriction of the hyperplane class from $\mathbb{P}^4$; this lattice admits an involution fixing K_S with $H \mapsto 4K_S - H$. See [Ran88, Proposition 9.18ff.], [Ran91], and [Bak10, p. 280] for discussion of these surfaces.

The Crauder-Katz classification therefore precludes diagrams of the form (5). We therefore recast our search as follows:

Question 32 Does there exist a diagram

$$\begin{tikzcd} & X \arrow[dl, "\beta_1"'] \arrow[dr, "\beta_2"] & \\ \mathbb{P}^4 & & \mathbb{P}^4 \end{tikzcd}$$

where X is smooth and the β_i are birational projective morphisms, and K3 surfaces S_1 and S_2 such that

- S_1 and S_2 are derived equivalent but not isomorphic;
- S_1 is birational to a center of β_1 but not to any center of β_2;
- S_2 is birational to a center of β_2 but not to any center of β_1?

This could yield counterexamples to the Larsen-Lunts cut-and-paste question on Grothendieck groups, similar to those found by Borisov [Bor14]. Galkin and Shinder [GS14, Sect. 7] showed that *if* the class of the affine line were a non-zero divisor in the Grothendieck group then for each rational cubic fourfold X the variety of lines $F_1(X)$ would be birational to $S^{[2]}$ for some K3 surface S. (Note that Borisov shows it *is* a zero divisor.) We have seen (Theorem 29) that this condition holds for infinitely many d.

3.7 Interpreting Associated K3 Surfaces II

Putting Theorem 29 on a general footing requires a larger inventory of varieties arising from K3 surfaces. We shall use fundamental results on moduli spaces of sheaves on K3 surfaces due to Mukai [Muk87], Yoshioka, and others. Let S be a complex projective K3 surface. The *Mukai lattice*

$$\tilde{H}^*(S,\mathbb{Z}) = H^0(S,\mathbb{Z})(-1) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})(1)$$

with unimodular form

$$((r_1, D_1, s_1), (r_2, D_2, s_2)) = -r_1 s_2 + D_1 D_2 - r_2 s_1$$

carries the structure of a Hodge structure of weight two. (The zeroth and fourth cohomology are of type $(1,1)$ and the middle cohomology carries its standard Hodge structure.) Thus we have

$$\tilde{H}^*(S,\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus_{\perp} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Suppose $v = (r, D, s) \in \tilde{H}^*(S,\mathbb{Z})$ is primitive of type $(1,1)$ with $(v, v) \geq 0$. Assume that one of the following holds:

- $r > 0$;
- $r = 0$ and D is ample.

Fixing a polarization h on S, we may consider the moduli space $M_v(S)$ of sheaves Gieseker stable with respect to h. Here r is the rank, D is the first Chern class, and $r + s$ is the Euler characteristic. For h chosen suitably general (see

[Yos01, Sect. 0] for more discussion), $M_v(S)$ is a projective holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of length $\frac{(v,v)}{2}+1$ zero dimensional subschemes of a K3 surface [Yos01, Sect. 8], [Yos00, Theorem 0.1]. Thus $H^2(M_v(S),\mathbb{Z})$ comes with a Beauville-Bogomolov form and we have an isomorphism of Hodge structures

$$H^2(M_v(S),\mathbb{Z}) = \begin{cases} v^{\perp}/\mathbb{Z}v & \text{if } (v,v)=0 \\ v^{\perp} & \text{if } (v,v)>0 \end{cases} \tag{6}$$

Example 33 The case of ideal sheaves of length two subschemes is $r=1, D=0$, and $s=-1$. Here we recover formula (3) for $H^2(S^{[2]},\mathbb{Z}) = H^2(M_{(1,0,-1)}(S),\mathbb{Z})$.

We shall also need recent results of Bayer-Macrì [BM14b, BM14a]: Suppose that M is holomorphic symplectic and birational to $M_v(S)$ for some K3 surface S. Then we may interpret M as a moduli space of objects on the derived category of S with respect to a suitable Bridgeland stability condition [BM14a, Theorem 1.2].

Finally, recall Nikulin's approach to lattice classification and embeddings [Nik79]. Given an even unimodular lattice Λ and a primitive nondegenerate sublattice $N \subset \Lambda$, the discriminant group $d(N) := N^*/N$ is equipped with a $(\mathbb{Q}/2\mathbb{Z})$-valued quadratic form q_N, which encodes most of the p-adic invariants of N. The orthogonal complement $N^{\perp} \subset \Lambda$ has related invariants

$$(d(N^{\perp}), -q_{N^{\perp}}) \simeq (d(N), q_N).$$

Conversely, given a pair of nondegenerate even lattices with complementary invariants, there exists a unimodular even lattice containing them as orthogonal complements [Nik79, Sect. 12].

Theorem 34 *Let (X, K_d) denote a labelled special cubic fourfold of discriminant d. Then d is admissible if and only if there exists a polarized K3 surface (S,f), a Mukai vector $v=(r,af,s)\in \tilde{H}(S,\mathbb{Z})$, a stability condition σ, and an isomorphism*

$$\varpi : M_v(S) \xrightarrow{\sim} F_1(X)$$

from the moduli space of objects in the derived category stable with respect to σ, inducing an isomorphism between the primitive cohomology of (S,f) and the twist of the non-special cohomology of (X, K_d).

This is essentially due to Addington and Thomas [AT14, Add16].

Proof Let's first do the reverse direction; this gives us an opportunity to unpack the isomorphisms in the statement. Assume we have the moduli space and isomorphism as described. After perhaps applying a shift and taking duals, we may assume $r \geq 0$ and $a \geq 0$; if $a = 0$ then $v = (1, 0, -1)$, i.e., the Hilbert scheme up to birational equivalence. We still have the computation (6) of the cohomology

$$H^2(M_v(S), \mathbb{Z}) = v^{\perp} \subset \tilde{H}^*(S, \mathbb{Z});$$

see [BM14b, Theorem 6.10] for discussion relating moduli spaces of Bridgeland-stable objects and Gieseker-stable sheaves. Thus we obtain a saturated embedding of the primitive cohomology of (S, f)

$$\Lambda_d \hookrightarrow H^2(M_v(S), \mathbb{Z}).$$

The isomorphism ϖ allows us to identify this with a sublattice of $H^2(F(X), \mathbb{Z})$ coinciding with $\alpha(K_d)^{\perp}$. Basic properties of the Abel-Jacobi map (1) imply that (S, f) is associated with (X, K_d), thus d is admissible by Theorem 14.

Now assume d is admissible and consider the lattice $-K_d^{\perp}$, the orthogonal complement of K_d in the middle cohomology of a cubic fourfold with the intersection form reversed. This is an even lattice of signature $(2, 19)$.

If X is a cubic fourfold then there is a natural primitive embedding of lattices [Mar11, Sect. 9]

$$H^2(F_1(X), \mathbb{Z}) \hookrightarrow \Lambda$$

where Λ is isomorphic to the Mukai lattice of a K3 surface

$$\Lambda = U^{\oplus 4} \oplus (-E_8)^{\oplus 2}.$$

Here 'natural' means that the monodromy representation on $H^2(F_1(X), \mathbb{Z})$ extends naturally to Λ.

Now consider the orthogonal complement M_d to $-K_d^{\perp}$ *in the Mukai lattice* Λ. Since d is admissible

$$-K_d^{\perp} \simeq \Lambda_d \simeq (-d) \oplus U^{\oplus 2} \oplus (-E_8)^{\oplus 2}$$

so $d(-K_d^{\perp}) \simeq \mathbb{Z}/d\mathbb{Z}$ and $q_{-K_d^{\perp}}$ takes value $-\frac{1}{d} \pmod{2\mathbb{Z}}$ on one of the generators. Thus $d(M_d) = \mathbb{Z}/d\mathbb{Z}$ and takes value $\frac{1}{d}$ on one of the generators. There is a distinguished lattice with these invariants

$$(d) \oplus U.$$

Kneser's Theorem [Nik79, Sect. 13] implies there is a unique such lattice, i.e., $M_d \simeq (d) \oplus U$.

Thus for each generator $\gamma \in d(-K_d^{\perp})$ with $(\gamma, \gamma) = -\frac{1}{d} \pmod{2\mathbb{Z}}$, we obtain an isomorphism of Hodge structures

$$H^2(F_1(X), \mathbb{Z}) \subset \Lambda \simeq \tilde{H}^*(S, \mathbb{Z})$$

where (S,f) is a polarized K3 surface of degree d. Here we take f to be one of generators of $U^{\perp} \subset M_d$. Let $v \in \Lambda$ generate the orthogonal complement to $H^2(F_1(X), \mathbb{Z})$; it follows that $v = (r, af, s) \in \tilde{H}^*(S, \mathbb{Z})$ and after reversing signs we may assume $r \geq 0$.

Consider a moduli space $M_v(S)$ of sheaves stable with respect to suitably generic polarizations on S. Our lattice analysis yields

$$\phi : H^2(F_1(X), \mathbb{Z}) \xrightarrow{\sim} H^2(M_v(S), \mathbb{Z}),$$

an isomorphism of Hodge structures taking $\alpha(K_d^{\perp})$ to the primitive cohomology of S. The Torelli Theorem [Mar11, Corollary 9.9] yields a birational map

$$\varpi_1 : M_v(S) \overset{\sim}{\dashrightarrow} F_1(X);$$

since both varieties are holomorphic symplectic, there is a natural induced isomorphism [Huy99, Lemma 2.6]

$$\varpi_1^* : H^2(F_1(X), \mathbb{Z}) \xrightarrow{\sim} H^2(M_v(S), \mathbb{Z}),$$

compatible with Beauville-Bogomolov forms and Hodge structures.

A priori ϕ and ϖ_1^* might differ by an automorphism of the cohomology of $M_v(S)$. If this automorphism permutes the two connected components of the positive cone in $H^2(M_v(S), \mathbb{R})$, we may reverse the sign of ϕ. If it fails to preserve the moving cone, we can apply a sequence of monodromy reflections on $M_v(S)$ until this is the case [Mar11, Theorems 1.5, 1.6]. These are analogues to reflections by (-2)-classes on the cohomology of K3 surfaces and are explicitly known for manifolds deformation equivalent to Hilbert schemes on K3 surface; see [HT01, HT09] for the case of dimension four and [Mar11, Sect. 9.2.1] for the general picture. In this situation, the reflections correspond to spherical objects in the derived category of S orthogonal to v, thus give rise to autoequivalences on the derived category [ST01, HLOY04]. We use these to modify the stability condition on $M_v(S)$. The resulting

$$\varpi_2^* : H^2(F_1(X), \mathbb{Z}) \xrightarrow{\sim} H^2(M_v(S), \mathbb{Z})$$

differs from ϕ by an automorphism that preserves moving cones, but may not preserve polarizations. Using [BM14a, Theorem 1.2(b)], we may modify the stability condition on $M_v(S)$ yet again so that the polarization g on $F_1(X)$ is identified with a polarization on $M_v(S)$. Then the resulting

$$\varpi = \varpi_3 : M_v(S) \overset{\sim}{\dashrightarrow} F_1(X);$$

preserves ample cones and thus is an isomorphism. Hence $F_1(X)$ is isomorphic to some moduli space of σ-stable objects over S. $\square$

Remark 35 As suggested by Addington and Thomas [AT14, Sect. 7.4], it should be possible to employ stability conditions to show that $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface if and only if X admits an associated K3 surface. (We use the notation of Sect. 3.3.)

Remark 36 Two K3 surfaces S_1 and S_2 are derived equivalent if and only if

$$\tilde{H}(S_1, \mathbb{Z}) \simeq \tilde{H}(S_2, \mathbb{Z})$$

as weight two Hodge structures [Orl97, Sect. 3]. The proof of Theorem 34 explains why the K3 surfaces associated with a given cubic fourfold are all derived equivalent, as mentioned in Remark 27.

There are other geometric explanations for K3 surfaces associated with special cubic fourfolds. Fix X to be a cubic fourfold not containing a plane. Let $M_3(X)$ denote the moduli space of generalized twisted cubics on X, i.e., closed subschemes arising as flat limits of twisted cubics in projective space. Then $M_3(X)$ is smooth and irreducible of dimension ten [LLSvS15, Theorem A]. Choose $[C] \in M_3(X)$ such that C is a smooth twisted cubic curve and $W := \mathrm{span}(C) \cap X$ is a smooth cubic surface. Then the linear series $|\mathcal{O}_W(C)|$ is two-dimensional, so we have a distinguished subvariety

$$[C] \in \mathbb{P}^2 \subset M_3(X).$$

Then there exists an eight-dimensional irreducible holomorphic symplectic manifold Z and morphisms

$$M_3(X) \xrightarrow{a} Z' \xrightarrow{\sigma} Z$$

where a is an étale-locally trivial $\mathbb{P}^2$ bundle and σ is birational [LLSvS15, Theorem B]. Moreover, Z is deformation equivalent to the Hilbert scheme of length four subschemes on a K3 surface [AL15]. Indeed, if X is Pfaffian with associated K3 surface S then Z is birational to $S^{[4]}$. It would be useful to have a version of Theorem 34 with Z playing the role of $F_1(X)$.

4 Unirational Parametrizations

We saw in Sect. 1.7 that smooth cubic fourfolds always admit unirational parametrizations of degree two. How common are unirational parametrizations of odd degree?

We review the double point formula [Ful84, Sect. 9.3]: Let S' be a nonsingular projective surface, P a nonsingular projective fourfold, and $f : S' \to P$ a morphism with image $S = f(S')$. We assume that $f : S' \to S$ is an isomorphism away from a finite subset of S'; equivalently, S has finitely many singularities with normalization

$f : S' \to S$. The *double point class* $\mathbb{D}(f) \in \mathrm{CH}_0(S')$ is given by the formula

$$\begin{aligned}\mathbb{D}(f) &= f^*f_*[S'] - (c(f^*T_P)c(T_{S'})^{-1})_2 \cap [S'] \\ &= f^*f_*[S'] - (c_2(f^*T_P) - c_1(T_{S'})c_1(f^*T_P) + c_1(T_{S'})^2 - c_2(T_{S'}))\end{aligned}$$

We define

$$D_{S\subset P} = \frac{1}{2}([S]_P^2 - c_2(f^*T_P) + c_1(T_{S'})c_1(f^*T_P) - c_1(T_{S'})^2 + c_2(T_{S'}));$$

if S has just transverse double points then $D_{S\subset P}$ is the number of these singularities.

Example 37 (cf. Lemma 3) Let $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ be a quartic scroll, $P = \mathbb{P}^4$, and $f : S' \to \mathbb{P}^4$ a generic projection. Then we have

$$D_{S\subset\mathbb{P}^4} = \frac{1}{2}(16 - 40 + 30 - 8 + 4) = 1$$

double point.

Proposition 38 *Let X be a cubic fourfold and $S \subset X$ a rational surface of degree d. Suppose that S has isolated singularities and smooth normalization S', with invariants $D = \deg(S)$, section genus g, and self-intersection $\langle S, S\rangle$. If*

$$\varrho = \varrho(S, X) := \frac{D(D-2)}{2} + (2 - 2g) - \frac{1}{2}\langle S, S\rangle > 0 \tag{7}$$

then X admits a unirational parametrization $\rho : \mathbb{P}^4 \dashrightarrow X$ of degree ϱ.

This draws on ideas from [HT01, Sect. 7] and [Voi14, Sect. 5].

Proof We analyze points $x \in X$ such that the projection

$$f = \pi_x : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

maps S birationally to a surface $\hat{S}$ with finitely many singularities, and $f : S \to \hat{S}$ is finite and unramified. Thus S and $\hat{S}$ have the same normalization.

Consider the following conditions:

1. x is contained in the tangent space to some singular point $s \in S$;
2. x is contained in the tangent space to some smooth point $s \in S$;
3. x is contained in a positive-dimensional family of secants to S.

The first condition can be avoided by taking x outside a finite collection of linear subspaces. The second condition can be avoided by taking x outside the tangent variety of S. This cannot coincide with a smooth cubic fourfold, which contains at most finitely many two-planes [BHB06, Appendix]. We turn to the third condition. If the secants to S sweep out a subvariety $Y \subsetneq \mathbb{P}^5$ then Y cannot be a smooth cubic fourfold. (The closure of Y contains all the tangent planes to S.) When the secants to S dominate $\mathbb{P}^5$ then the locus of points in $\mathbb{P}^5$ on infinitely many secants has codimension at least two.

Projection from a point on X outside these three loci induces a morphism

$$S \to \hat{S} = f(S)$$

birational and unramified onto its image. Moreover, this image has finitely many singularities that are resolved by normalization.

Let W denote the second symmetric power of S. Since S is rational, W is rational as well. There is a rational map coming from residual intersection

$$\begin{aligned} \rho : W &\dashrightarrow X \\ s_1 + s_2 &\mapsto x \end{aligned}$$

where $\ell(s_1, s_2) \cap X = \{s_1, s_2, x\}$. This is well-defined at the generic point of W as the generic secant to S is not contained in X. (An illustrative special case can be found in Remark 5.)

The degree of ρ is equal to the number of secants to S through a generic point of X. The analysis above shows this equals the number of secants to S through a generic point of $\mathbb{P}^5$. These in turn correspond to the number of double points of $\hat{S}$ arising from *generic* projection to $\mathbb{P}^4$, i.e.,

$$\deg(\rho) = D_{\hat{S}\subset\mathbb{P}^4} - D_{S\subset X}.$$

The double point formula gives

$$\begin{aligned} 2D_{S,X} &= \langle S, S\rangle - (c_2(T_X|S') - c_1(T_{S'})c_1(T_X|S') + c_1(T_{S'}) - c_2(T_{S'})) \\ 2D_{\hat{S},\mathbb{P}^4} &= \left\langle \hat{S}, \hat{S}\right\rangle_{\mathbb{P}^4} - (c_2(T_{\mathbb{P}^4}|S') - c_1(T_{S'})c_1(T_{\mathbb{P}^4}|S') + c_1(T_{S'}) - c_2(T_{S'})) \end{aligned}$$

where $\left\langle \hat{S}, \hat{S}\right\rangle_{\mathbb{P}^4} = D^2$ by Bezout's Theorem. Taking differences (cf. Sect. 2.2) yields

$$D_{\hat{S}\subset\mathbb{P}^4} - D_{S\subset X} = \frac{1}{2}(D^2 - 4D + 2Hc_1(T_{S'}) + \langle S, S\rangle),$$

where $H = h|S$. Using the adjunction formula

$$2g - 2 = H^2 + K_{S'}H$$

we obtain (7). □

Corollary 39 (Odd Degree Unirational Parametrizations) *Retain the notation of Proposition 38 and assume that S is not homologous to a complete intersection. Consider the discriminant*

$$d = 3\,\langle S, S\rangle - D^2$$

of

$$\begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & D \\ S & D & \langle S, S\rangle \end{array}.$$

Then the degree

$$\varrho(S, X) = \frac{d}{2} - 2\langle S, S\rangle + (2-2g) + (D^2 - D)$$

has the same parity as $\frac{d}{2}$. *Thus the degree of* $\rho : \mathbb{P}^4 \dashrightarrow X$ *is odd provided* d *is not divisible by four.*

Compare this with Theorem 45 below.

How do we obtain surfaces satisfying the assumptions of Proposition 38? Nuer [Nue15, Sect. 3], [Nue17] exhibits *smooth* such surfaces for all $d \leq 38$, thus we obtain

Corollary 40 *A generic cubic fourfold* $X \in \mathcal{C}_d$*, for* $d = 14, 18, 26, 30$*, and* 38*, admits a unirational parametrization of odd degree.*

There are heuristic constructions of such surfaces in far more examples [HT01, Sect. 7]. Let $X \in \mathcal{C}_d$ and consider its variety of lines $F_1(X)$; for simplicity, assume the Picard group of $F_1(X)$ has rank two. Recent work of [BHT15, BM14a] completely characterizes rational curves

$$R \simeq \mathbb{P}^1 \subset F_1(X)$$

associated with extremal birational contractions of $F_1(X)$. The incidence correspondence

$$\begin{tikzcd} \mathcal{INC} \arrow[r] \arrow[d] & F_1(X) \\ X & \end{tikzcd}$$

yields

$$S' := \mathcal{INC}|R \to S \subset X,$$

i.e., a ruled surface with smooth normalization.

Question 41 When does the resulting ruled surfaces have isolated singularities? Is this the case when R is a generic rational curve arising as an extremal ray of a birational contraction?

The discussion of [HT01, Sect. 7] fails to address the singularity issues, and should be seen as a heuristic approach rather than a rigorous construction. The technical issues are illustrated by the following:

Example 42 (Voisin's Counterexample) Assume that X is not special so that $\mathrm{Pic}(F_1(X)) = \mathbb{Z}g$. Let R denote the positive degree generator of the Hodge classes $N_1(X,\mathbb{Z}) \subset H_2(F_1(X),\mathbb{Z})$; the lattice computations in Sect. 3.5 imply that

$$g \cdot R = \frac{1}{2}(g,g) = 3.$$

Moreover, there is a two-parameter family of rational curves $\mathbb{P}^1 \subset X, [\mathbb{P}^1] = R$, corresponding to the cubic surfaces $S' \subset X$ singular along a line.

These may be seen as follows: The cubic surfaces singular along some line have codimension seven in the parameter space of all cubic surfaces. However, there is a nine-parameter family of cubic surface sections of a given cubic fourfold, parametrized by $\mathrm{Gr}(4,6)$. Indeed, for a fixed flag

$$\ell \subset \mathbb{P}^3 \subset \mathbb{P}^5$$

a tangent space computation shows that the cubic fourfolds

$$\{X : X \cap \mathbb{P}^3 \text{ singular along the line } \ell\}$$

dominate the moduli space $\mathcal{C}$.

Let $S' \subset \mathbb{P}^3$ be a cubic surface singular along a line and $X \supset S'$ a smooth cubic fourfold. Since the generic point of X does not lie on a secant line of S', it cannot be used to produce a unirational parametrization of X. The reasoning for Proposition 38 and formula (7) is not valid, as S' has non-isolated singularities.

Nevertheless, the machinery developed here indicates where to look for unirational parameterizations of odd degree:

Example 43 ($d = 42$ Case) Let $X \in \mathcal{C}_{42}$ be generic. By Theorem 29, $F_1(X) \simeq T^{[2]}$ where (T,f) is K3 surface of degree 42 and $g = 2f - 9\delta$. Take R to be one of the rulings of the divisor in $T^{[2]}$ parametrizing non-reduced subschemes, i.e., those subschemes supported at a prescribed point of T. Note that $R \cdot g = 9$ so the ruled surface S associated with the incidence correspondence has numerical invariants:

	h^2	S
h^2	3	9
S	9	41

Assuming S has isolated singularities, we have $D_{S\subset X} = 8$ and X admits a unirational parametrization of degree $\varrho(S \subset X) = 13$. **Challenge:** Verify the singularity assumption for some $X \in \mathcal{C}_{42}$. This has been addressed by K.W. Lai in 'New cubic fourfolds with odd degree unirational parametrizations' arXiv:1606.03853.

5 Decomposition of the Diagonal

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. A *decomposition of the diagonal* of X is a rational equivalence in $\mathrm{CH}^n(X \times X)$

$$N\Delta_X \equiv N\{x \times X\} + Z,$$

where N is a non-zero integer, $x \in X(\mathbb{C})$, and Z is supported on $X \times D$ for some subvariety $D \subsetneq X$. Let $N(X)$ denote the smallest positive integer N occurring in a decomposition of the diagonal, which coincides with the greatest common divisor of such integers. If $N(X) = 1$ we say X admits an *integral decomposition of the diagonal*.

Proposition 44 ([ACTP13, Lemma 1.3] [Voi15a, Lemma 4.6]) *X admits a decomposition of the diagonal if and only if*

$$A_0 = \{P \in \mathrm{CH}_0 : \deg(P) = 0\}$$

is universally N-torsion for some positive integer N, i.e., for each extension $F/\mathbb{C}$ we have $NA_0(X_F) = 0$. Moreover, $N(X)$ is the annihilator of the torsion.

We sketch this for the convenience of the reader: Basic properties of Chow groups give the equivalence of the decomposition of $N\Delta_X$ with $NA_0(X_{\mathbb{C}(X)}) = 0$. Indeed, $A_0(X_{\mathbb{C}(X)})$ is the inverse limit of $A_0(X \times U)$ for all open $U \subset X$. Conversely, taking the basechange of a decomposition of the diagonal to the extension F gives that $A_0(X_F)$ is annihilated by N.

We recall situations where we have decompositions of the diagonal:

Rationally Connected Varieties Suppose X is rationally connected and choose $\beta \in H_2(X, \mathbb{Z})$ such that the evaluation

$$M_{0,2}(X, \beta) \to X \times X$$

is dominant. Fix an irreducible component M of $M_{0,2}(X, \beta) \times_{X \times X} \mathbb{C}(X \times X)$. Then $N(X)$ divides the index $\iota(M)$. Indeed, each effective zero-cycle $Z \subset M$ corresponds to $|Z|$ conjugate rational curves joining generic $x_1, x_2 \in X$. Together these give $|Z|x_1 = |Z|x_2$ in $\mathrm{CH}_0(X_{\mathbb{C}(X)})$. Thus we obtain a decomposition of the diagonal. See [CT05, Proposition 11] for more details.

Unirational Varieties If $\rho : \mathbb{P}^n \dashrightarrow X$ has degree ϱ then $N(X)|\varrho$. Thus $N(X)$ divides the greatest common divisor of the degrees of unirational parametrizations of X. A cubic hypersurface X of dimension at least two admits a degree two unirational parametrization (see Prop 10), so $N(X)|2$. We saw in Sect. 4 that many classes of special cubic fourfolds admit odd degree unirational parametrizations. In these cases, we obtain *integral* decompositions of the diagonal.

Rational and Stably Rational Varieties The case of rational varieties follows from our analysis of unirational parametrizations. For the stably rational case, it suffices to observe that

$$A_0(Y) \simeq A_0(Y \times \mathbb{P}^1)$$

and use the equivalence of Proposition 44 (see [Voi15a, Proposition 4.7] for details). Here we obtain an *integral* decomposition of the diagonal.

Remarkably, at least half of the special cubic fourfolds admit integral decompositions of the diagonal:

Theorem 45 ([Voi14, Theorem 5.6]) *A special cubic fourfold of discriminant $d \equiv 2 \pmod 4$ admits an integral decomposition of the diagonal.*

This suggests the following question:

Question 46 Do special cubic fourfolds of discriminant $d \equiv 2 \pmod 4$ always admit unirational parametrizations of odd degree? Are they stably rational?

Cubic fourfolds do satisfy a universal cohomological condition that follows from an integral decomposition of the diagonal: They admit universally trivial unramified H^3. This was proved first for cubic fourfolds containing a plane ($d = 8$) using deep properties of quadratic forms [ACTP13], then in general by Voisin [Voi15b, Example 3.2].

Question 47 Is there a cubic fourfold X with

$$K_d = H^{2,2}(X, \mathbb{Z}), \quad d \equiv 0 \pmod 4,$$

and admitting an integral decomposition of the diagonal? A unirational parametrization of odd degree?

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Derived Categories View on Rationality Problems

Alexander Kuznetsov

Abstract We discuss a relation between the structure of derived categories of smooth projective varieties and their birational properties. We suggest a possible definition of a birational invariant, the derived category analogue of the intermediate Jacobian, and discuss its possible applications to the geometry of prime Fano threefolds and cubic fourfolds.

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1 Introduction into Derived Categories and Semiorthogonal Decompositions

Derived categories were defined by Verdier in his thesis [Ver65] back in the 1960s. When they appeared they were used as an abstract notion to formulate general results, like Grothendieck–Riemann–Roch theorem (for which actually they were devised by Grothendieck). Later on, they were actively used as a technical tool, see e.g. [Har66]. The situation changed with appearance of Beilinson's brilliant paper [Bei78], when they attracted attention as the objects of investigation. Finally, results of Bondal and Orlov [BO95, BO02] put them on their present place in the center of algebraic geometry.

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1.1 Derived Categories

We refer to [GM99, Chap. III, IV] for a classical treatment of derived and triangulated categories, and to [H06] for a more geometrically oriented discussion. Here we restrict ourselves to give a very brief introduction into the subject.

Let k be a base field. Recall that a complex over a k-linear abelian category $\mathcal{A}$ is a collection of objects $F^i \in \mathcal{A}$ and morphisms $d_F^i : F^i \to F^{i+1}$ such that $d_F^{i+1} \circ d_F^i = 0$ for all $i \in \mathbb{Z}$. A complex is bounded if $F^i = 0$ for all $|i| \gg 0$. A morphism of complexes $(F^\bullet, d_F^\bullet) \to (G^\bullet, d_G^\bullet)$ is a collection of morphisms $\varphi^i : F^i \to G^i$ in $\mathcal{A}$ commuting with the differentials: $d_G^i \circ \varphi^i = \varphi^{i+1} \circ d_F^i$ for all $i \in \mathbb{Z}$. The category of bounded complexes in $\mathcal{A}$ is denoted by $\mathsf{Com}^b(\mathcal{A})$.

The ith cohomology of a complex $(F^\bullet, d_F^\bullet)$ is an object of the abelian category $\mathcal{A}$ defined by

$$\mathcal{H}^i(F) = \frac{\mathsf{Ker}(d_F^i : F^i \to F^{i+1})}{\mathsf{Im}(d_F^{i-1} : F^{i-1} \to F^i)}.$$

Clearly, the cohomology is a functor $\mathcal{H}^i : \mathsf{Com}^b(\mathcal{A}) \to \mathcal{A}$. In particular, a morphism of complexes $\varphi : F^\bullet \to G^\bullet$ induces a morphism of cohomology objects $\mathcal{H}^i(\varphi) : \mathcal{H}^i(F^\bullet) \to \mathcal{H}^i(G^\bullet)$. A morphism $\varphi : F^\bullet \to G^\bullet$ of complexes is called a quasiisomorphism if for all $i \in \mathbb{Z}$ the morphism $\mathcal{H}^i(\varphi)$ is an isomorphism.

Definition 1.1 The bounded derived category of an abelian category $\mathcal{A}$ is the localization

$$\mathbf{D}^b(\mathcal{A}) = \mathsf{Com}^b(\mathcal{A})[\mathsf{Qiso}^{-1}].$$

of the category of bounded complexes in $\mathcal{A}$ with respect to the class of quasiisomorphisms.

Of course, this definition itself requires an explanation, which we prefer to skip since there are many textbooks (e.g. [GM99]) describing this in detail. Here we just restrict ourselves by saying that a localization of a category $\mathcal{C}$ in a class of morphisms S is a category $\mathcal{C}[S^{-1}]$ with a functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ such that the images of all morphisms in S under this functor are invertible, and which is the smallest with this property (that is enjoys a universal property).

Sometimes it is more convenient to consider the unbounded version of the derived category, but we will not need it.

The cohomology functors descend to the derived category, so $\mathcal{H}^i : \mathbf{D}^b(\mathcal{A}) \to \mathcal{A}$ is an additive functor for each $i \in \mathbb{Z}$. Further, there is a full and faithful embedding functor

$$\mathcal{A} \to \mathbf{D}^b(\mathcal{A}),$$

taking an object $F \in \mathcal{A}$ to the complex $\cdots \to 0 \to F \to 0 \to \ldots$ with zeroes everywhere outside of degree zero, in which the object F sits. This complex has only one nontrivial cohomology which lives in degree zero and equals F.

1.2 Triangulated Categories

The most important structure on the derived category is the triangulated structure.

Definition 1.2 A triangulated category is an additive category $\mathcal{T}$ equipped with

- an automorphism of $\mathcal{T}$ called the shift functor and denoted by $[1] : \mathcal{T} \to \mathcal{T}$, the powers of the shift functor are denoted by $[k] : \mathcal{T} \to \mathcal{T}$ for all $k \in \mathbb{Z}$;
- a class of chains of morphisms in $\mathcal{T}$ of the form

$$F_1 \xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} F_1[1] \tag{1}$$

  called distinguished triangles,

which satisfy a number of axioms (see [GM99]).

Instead of listing all of them we will discuss only the most important axioms and properties.

First, each morphism $F_1 \xrightarrow{\varphi_1} F_2$ can be extended to a distinguished triangle (1). The extension is unique up to a noncanonical isomorphism, the third vertex of such a triangle is called a cone of the morphism φ_1 and is denoted by $\mathsf{Cone}(\varphi_1)$.

Further, a triangle (1) is distinguished if and only if the triangle

$$F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} F_1[1] \xrightarrow{-\varphi_1[1]} F_2[1] \tag{2}$$

is distinguished. Such triangle is referred to as the rotation of the original triangle. Clearly, rotating a distinguished triangle in both directions, one obtains an infinite chain of morphisms

$$\ldots \to F_3[-1] \xrightarrow{-\varphi_3[-1]} F_1$$
$$\xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} F_1[1] \xrightarrow{-\varphi_1[1]} F_2[1] \xrightarrow{-\varphi_2[1]} F_3[1] \xrightarrow{-\varphi_3[1]} F_1[2] \to \ldots,$$

called a helix. Any consecutive triple of morphisms in a helix is thus a distinguished triangle.

Finally, the sequence of k-vector spaces

$$\dots \to \mathsf{Hom}(G, F_3[-1]) \xrightarrow{-\varphi_3[-1]} \mathsf{Hom}(G, F_1) \xrightarrow{\varphi_1} \mathsf{Hom}(G, F_2) \xrightarrow{\varphi_2} \mathsf{Hom}(G, F_3) \xrightarrow{\varphi_3} \mathsf{Hom}(G, F_1[1]) \to \dots,$$

obtained by applying the functor $\mathsf{Hom}(G, -)$ to a helix, is a long exact sequence. Analogously, the sequence of k-vector spaces

$$\dots \to \mathsf{Hom}(F_1[1], G) \xrightarrow{\varphi_3} \mathsf{Hom}(F_3, G) \xrightarrow{\varphi_2} \mathsf{Hom}(F_2, G) \xrightarrow{\varphi_1} \mathsf{Hom}(F_1, G) \xrightarrow{-\varphi_3[-1]} \mathsf{Hom}(F_3[-1], G) \to \dots,$$

obtained by applying the functor $\mathsf{Hom}(-, G)$ to a helix, is a long exact sequence.

Exercise 1.3 Assume that in a distinguished triangle (1) one has $\varphi_3 = 0$. Show that there is an isomorphism $F_2 \cong F_1 \oplus F_3$ such that φ_1 is the embedding of the first summand, and φ_2 is the projection onto the second summand.

Exercise 1.4 Show that a morphism $\varphi_1 : F_1 \to F_2$ is an isomorphism if and only if $\mathsf{Cone}(\varphi_1) = 0$.

Derived category $\mathbf{D}^b(\mathcal{A})$ carries a natural triangulated structure. The shift functor is defined by

$$(F[1])^i = F^{i+1}, \qquad d^i_{F[1]} = -d^{i+1}_F, \qquad (\varphi[1])^i = \varphi^{i+1}. \tag{3}$$

The cone of a morphism of complexes $\varphi : F \to G$ is defined by

$$\mathsf{Cone}(\varphi)^i = G^i \oplus F^{i+1}, \qquad d_{\mathsf{Cone}(\varphi)}(g^i, f^{i+1}) = (d_G(g^i) + \varphi(f^{i+1}), -d_F(f^{i+1})). \tag{4}$$

A morphism $F \xrightarrow{\varphi} G$ extends to a triangle by morphisms

$$\epsilon : G \to \mathsf{Cone}(\varphi), \quad \epsilon(g^i) = (g^i, 0), \qquad \rho : \mathsf{Cone}(\varphi) \to F[1], \quad \rho(g^i, f^{i+1}) = f^{i+1}.$$

One defines a distinguished triangle in $\mathbf{D}^b(\mathcal{A})$ as a triangle isomorphic to the triangle

$$F \xrightarrow{\varphi} G \xrightarrow{\epsilon} \mathsf{Cone}(\varphi) \xrightarrow{\rho} F[1]$$

defined above.

Theorem 1.5 ([Ver65, Ver96]) *The shift functor and the above class of distinguished triangles provide $\mathbf{D}^b(\mathcal{A})$ with a structure of a triangulated category.*

For $F, G \in \mathcal{A}$ the spaces of morphisms in the derived category between F and shifts of G are identified with the Ext-groups in the original abelian category

$$\mathsf{Hom}(F, G[i]) = \mathsf{Ext}^i(F, G).$$

For arbitrary objects $F, G \in \mathbf{D}^b(\mathcal{A})$ we use the left hand side of the above equality as the definition of the right hand side. With this convention the long exact sequences obtained by applying Hom functors to a helix can be rewritten as

$$\dots \to \mathsf{Ext}^{i-1}(G, F_3) \to \mathsf{Ext}^i(G, F_1) \to \mathsf{Ext}^i(G, F_2) \to \mathsf{Ext}^i(G, F_3) \to \mathsf{Ext}^{i+1}(G, F_1) \to \dots$$

$$\dots \to \mathsf{Ext}^{i-1}(F_1, G) \to \mathsf{Ext}^i(F_3, G) \to \mathsf{Ext}^i(F_2, G) \to \mathsf{Ext}^i(F_1, G) \to \mathsf{Ext}^{i+1}(F_3, G) \to \dots$$

The most important triangulated category for the geometry of an algebraic variety X is the bounded derived category of coherent sheaves on X. To make notation more simple we use the following shorthand

$$\mathbf{D}(X) := \mathbf{D}^b(\mathsf{coh}(X)).$$

Although most of the results we will discuss are valid in a much larger generality, we restrict for simplicity to the case of smooth projective varieties. Sometimes one also may need some assumptions on the base field, so let us assume for simplicity that $\mathsf{k} = \mathbb{C}$.

1.3 Functors

A triangulated functor between triangulated categories $\mathcal{T}_1 \to \mathcal{T}_2$ is a pair (Φ, ϕ), where $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ is a k-linear functor $\mathcal{T}_1 \to \mathcal{T}_2$, which takes distinguished triangles of $\mathcal{T}_1$ to distinguished triangles of $\mathcal{T}_2$, and $\phi : \Phi \circ [1]_{\mathcal{T}_1} \to [1]_{\mathcal{T}_2} \circ \Phi$ is an isomorphism of functors. Usually the isomorphism ϕ will be left implicit.

There is a powerful machinery (see [GM99] for the classical or [Kel06] for the modern approach) which allows to extend an additive functor between abelian categories to a triangulated functor between their derived categories. Typically, if the initial functor is right exact, one extends it as the left derived functor (by applying the original functor to a resolution of the object of projective type), and if the initial functor is left exact, one extends it as the right derived functor (by using a resolution of injective type). We do not stop here on this technique. Instead, we list the most important (from the geometric point of view) triangulated functors between derived categories of coherent sheaves (see [Har66], or [H06] for more details).

1.3.1 Pullbacks and Pushforwards

Let $f : X \to Y$ be a morphism of (smooth, projective) schemes. It gives an adjoint pair of functors (f^*, f_*), where

- $f^* : \mathsf{coh}(Y) \to \mathsf{coh}(X)$ is the pullback functor, and
- $f_* : \mathsf{coh}(X) \to \mathsf{coh}(Y)$ is the pushforward functor.

This adjoint pair induces an adjoint pair of derived functors (Lf^*, Rf_*) on the derived categories, where

- $Lf^* : \mathbf{D}(Y) \to \mathbf{D}(X)$ is the (left) derived pullback functor, and
- $Rf_* : \mathbf{D}(X) \to \mathbf{D}(Y)$ is the (right) derived pushforward functor.

The cohomology sheaves of the derived pullback/pushforward applied to a coherent sheaf $\mathcal{F}$ are well known as the classical higher pullbacks/pushforwards

$$L_i f^*(\mathcal{F}) = \mathcal{H}^{-i}(Lf^*\mathcal{F}), \qquad R^i f_*(\mathcal{F}) = \mathcal{H}^i(Rf_*\mathcal{F}).$$

When $f : X \to \mathsf{Spec}(\mathsf{k})$ is the structure morphim of a k-scheme X, the pushforward functor f_* is identified with the global sections functor $\Gamma(X, -)$, so that one has $R^i f_* = H^i(X, -)$ and $Rf_* \cong R\Gamma(X, -)$.

1.3.2 Twisted Pullbacks

The derived pushforward functor Rf_* also has a right adjoint functor

$$f^! : \mathbf{D}(Y) \to \mathbf{D}(X),$$

which is called sometimes the twisted pullback functor. The adjunction of Rf_* and $f^!$ is known as the Grothendieck duality. The twisted pullback $f^!$ has a very simple relation with the derived pullback functor (under our assumption of smoothness and projectivity)

$$f^!(F) \cong Lf^*(F) \otimes \omega_{X/Y}[\dim X - \dim Y],$$

where $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$, is the relative dualizing sheaf.

1.3.3 Tensor Products and Local $\mathscr{H}om$'s

Another important adjoint pair of functors on the category $\mathsf{coh}(X)$ of coherent sheaves is $(\otimes, \mathscr{H}om)$. In fact these are bifunctors (each has two arguments), and the adjunction is a functorial isomorphism

$$\mathsf{Hom}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) \cong \mathsf{Hom}(\mathcal{F}_1, \mathscr{H}om(\mathcal{F}_2, \mathcal{F}_3)).$$

This adjoint pair induces an adjoint pair of derived functors $(\overset{\mathbb{L}}{\otimes}, \mathsf{R}\mathscr{H}\!om)$ on the derived categories, where

- $\overset{\mathbb{L}}{\otimes} : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X)$ is the (left) derived tensor product functor, and
- $\mathsf{R}\mathscr{H}\!om : \mathbf{D}(X)^{\mathsf{opp}} \times \mathbf{D}(X) \to \mathbf{D}(X)$ is the (right) derived local $\mathscr{H}\!om$ functor.

The cohomology sheaves of these functors applied to a pair of coherent sheaves $\mathcal{F}, \mathcal{G} \in \mathsf{coh}(X)$ are the classical Tor's and $\mathcal{E}xt$'s:

$$\mathsf{Tor}_i(\mathcal{F}, \mathcal{G}) = \mathcal{H}^{-i}(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G}), \qquad \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = \mathcal{H}^i(\mathsf{R}\mathscr{H}\!om(\mathcal{F}, \mathcal{G})).$$

One special case of the $\mathsf{R}\mathscr{H}\!om$ functor is very useful. The object

$$F^\vee := \mathsf{R}\mathscr{H}\!om(F, \mathcal{O}_X)$$

is called the (derived) dual object. With the smoothness assumption there is a canonical isomorphism

$$\mathsf{R}\mathscr{H}\!om(F, G) \cong F^\vee \overset{\mathbb{L}}{\otimes} G$$

for all $F, G \in \mathbf{D}(X)$.

1.4 Relations

The functors we introduced so far obey a long list of relations. Here we discuss the most important of them.

1.4.1 Functoriality

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of morphisms. Then

$$R(g \circ f)_* \cong Rg_* \circ Rf_*,$$
$$L(g \circ f)^* \cong Lf^* \circ Lg^*,$$
$$(g \circ f)^! \cong f^! \circ g^!.$$

In particular, if $Z = \mathsf{Spec}\,\mathsf{k}$ is the point then the first formula gives an isomorphism $R\Gamma \circ Rf_* \cong R\Gamma$.

1.4.2 Local Adjunctions

There are isomorphisms

$$Rf_* \, \mathsf{R}\mathscr{H}om(Lf^*(F), G) \cong \mathsf{R}\mathscr{H}om(F, Rf_*(G)),$$
$$Rf_* \, \mathsf{R}\mathscr{H}om(G, f^!(F)) \cong \mathsf{R}\mathscr{H}om(Rf_*(G), F).$$

If one applies the functor $R\Gamma$ to these formulas, the usual adjunctions are recovered. Another local adjunction is the following isomorphism

$$\mathsf{R}\mathscr{H}om(F \overset{\mathbb{L}}{\otimes} G, H) \cong \mathsf{R}\mathscr{H}om(F, \mathsf{R}\mathscr{H}om(G, H)).$$

1.4.3 Tensor Products and Pullbacks

Derived tensor product is associative and commutative, thus $\mathbf{D}(X)$ is a tensor (symmetric monoidal) category. The pullback functor is a tensor functor, i.e.

$$Lf^*(F \overset{\mathbb{L}}{\otimes} G) \cong Lf^*(F) \overset{\mathbb{L}}{\otimes} Lf^*(G),$$
$$Lf^* \, \mathsf{R}\mathscr{H}om(F, G) \cong \mathsf{R}\mathscr{H}om(Lf^*(F), Lf^*(G)).$$

1.4.4 The Projection Formula

In a contrast with the pullback, the pushforward is not a tensor functor. It has, however, a weaker property

$$Rf_*(Lf^* F \overset{\mathbb{L}}{\otimes} G) \cong F \overset{\mathbb{L}}{\otimes} Rf_*(G),$$

which is called projection formula and is very useful. A particular case of it is the following isomorphism

$$Rf_*(Lf^* F) \cong F \overset{\mathbb{L}}{\otimes} Rf_*(\mathcal{O}_X).$$

1.4.5 Base Change

Let $f : X \to S$ and $u : T \to S$ be morphisms of schemes. Consider the fiber product $X_T := X \times_S T$ and the fiber square

$$\begin{array}{ccc} X_T & \xrightarrow{u_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{u} & S \end{array} \qquad (5)$$

Using adjunctions and functoriality of pullbacks and pushforwards, it is easy to construct a canonical morphism of functors $Lu^* \circ Rf_* \to Rf_{T*} \circ Lu_X^*$. The base change theorem says that it is an isomorphism under appropriate conditions. To formulate these we need the following

Definition 1.6 A pair of morphisms $f : X \to S$ and $u : T \to S$ is called Tor-independent if for all points $x \in X$, $t \in T$ such that $f(x) = s = u(t)$ one has

$$\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, \mathcal{O}_{T,t}) = 0 \qquad \text{for } i > 0.$$

Remark 1.7 If either f or u is flat then the square is Tor-independent. Furthermore, when X, S, and T are all smooth there is a simple sufficient condition for a pair (f, u) to be Tor-independent:

$$\dim X_T = \dim X + \dim T - \dim S,$$

i.e. the equality of the dimension of X_T and of its expected dimension.

Theorem 1.8 ([Li, Theorem 3.10.3]) *The base change morphism*

$$Lu^* \circ Rf_* \to Rf_{T*} \circ Lu_X^*$$

is an isomorphism if and only if the pair of morphisms $f : X \to S$ and $u : T \to S$ is Tor*-independent.*

1.5 Fourier–Mukai Functors

Let X and Y be algebraic varieties and $K \in \mathbf{D}(X \times Y)$ an object of the derived category of the product. Given this data we define a functor

$$\Phi_K : \mathbf{D}(X) \to \mathbf{D}(Y), \qquad F \mapsto Rp_{Y*}(K \overset{\mathbb{L}}{\otimes} Lp_X^*(F)),$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. It is called the Fourier–Mukai functor (other names are the kernel functor, the integral functor) with kernel K.

Fourier–Mukai functors form a nice class of functors, which includes most of the functors we considered before. This class is closed under compositions and adjunctions.

Exercise 1.9 Let $f : X \to Y$ be a morphism. Let $\gamma_f : X \to X \times Y$ be the graph of f. Show that the Fourier–Mukai functor with kernel $\gamma_{f*}\mathcal{O}_X \in \mathbf{D}(X \times Y)$ is isomorphic to the derived pushforward Rf_*.

Exercise 1.10 Let $g : Y \to X$ be a morphism. Let $\gamma_g : Y \to X \times Y$ be the graph of g. Show that the Fourier–Mukai functor with kernel $\gamma_{g*}\mathcal{O}_Y \in \mathbf{D}(X \times Y)$ is isomorphic to the derived pullback Lg^*.

Exercise 1.11 Let $\mathcal{E} \in \mathbf{D}(X)$ be an object. Let $\delta : X \to X \times X$ be the diagonal embedding. Show that the Fourier–Mukai functor with kernel $R\delta_*\mathcal{E} \in \mathbf{D}(X \times X)$ is isomorphic to the derived tensor product functor $\mathcal{E} \overset{\mathbb{L}}{\otimes} -$.

Exercise 1.12 Let $K_{12} \in \mathbf{D}(X_1 \times X_2)$ and $K_{23} \in \mathbf{D}(X_2 \times X_3)$. Consider the triple product $X_1 \times X_2 \times X_3$ and the projections $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$. Show an isomorphism of functors

$$\Phi_{K_{23}} \circ \Phi_{K_{12}} \cong \Phi_{K_{12} \circ K_{23}},$$

where $K_{12} \circ K_{23}$ is the convolution of kernels, defined by

$$K_{12} \circ K_{23} := Rp_{13*}(Lp_{12}^* K_{12} \overset{\mathbb{L}}{\otimes} Lp_{23}^* K_{23}).$$

Adjoint functors of Fourier–Mukai functors are also Fourier–Mukai functors.

Lemma 1.13 *The right adjoint functor of Φ_K is the Fourier–Mukai functor*

$$\Phi_K^! \cong \Phi_{K^\vee \overset{\mathbb{L}}{\otimes} \omega_X[\dim X]} : \mathbf{D}(Y) \to \mathbf{D}(X).$$

The left adjoint functor of Φ_K is the Fourier–Mukai functor

$$\Phi_K^* \cong \Phi_{K^\vee \overset{\mathbb{L}}{\otimes} \omega_Y[\dim Y]} : \mathbf{D}(Y) \to \mathbf{D}(X).$$

Proof The functor Φ_K is the composition of the derived pullback Lp_X^* , the derived tensor product with K, and the derived pushforward Rp_{Y*}. Therefore its right adjoint functor is the composition of their right adjoint functors, i.e. of the twisted pullback functor $p_Y^!$, the derived tensor product with $K^\vee$, and the derived pushforward functor Rp_{X*}. By Grothendieck duality we have $p_Y^!(G) \cong Lp_Y^*(G) \overset{\mathbb{L}}{\otimes} \omega_X[\dim X]$. Altogether, this gives the first formula. For the second part note that if $K' = K^\vee \overset{\mathbb{L}}{\otimes} \omega_Y[\dim Y]$ then the functor $\Phi_{K'} : \mathbf{D}(Y) \to \mathbf{D}(X)$ has a right adjoint, which by the first part of the Lemma coincides with Φ_K. Hence the left adjoint of Φ_K is $\Phi_{K'}$. □

1.6 Serre Functor

The notion of a Serre functor is a categorical interpretation of the Serre duality.

Definition 1.14 A Serre functor in a triangulated category $\mathcal{T}$ is an autoequivalence $\mathsf{S}_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ with a bifunctorial isomorphism

$$\mathsf{Hom}(F, G)^{\vee} \cong \mathsf{Hom}(G, \mathsf{S}_{\mathcal{T}}(F)).$$

It is easy to show (see [BK89]) that if a Serre functor exists, it is unique up to a canonical isomorphism. When $\mathcal{T} = \mathbf{D}(X)$, the Serre functor is given by a simple formula

$$\mathsf{S}_{\mathbf{D}(X)}(F) = F \otimes \omega_X[\dim X].$$

The bifunctorial isomorphism in its definition is the Serre duality for X.

If X is a Calabi–Yau variety (i.e. $\omega_X \cong \mathcal{O}_X$) then the corresponding Serre functor $\mathsf{S}_{\mathbf{D}(X)} \cong [\dim X]$ is just a shift. This motivates the following

Definition 1.15 ([K15b]) A triangulated category $\mathcal{T}$ is a Calabi–Yau category of dimension $n \in \mathbb{Z}$ if $\mathsf{S}_{\mathcal{T}} \cong [n]$. A triangulated category $\mathcal{T}$ is a fractional Calabi–Yau category of dimension $p/q \in \mathbb{Q}$ if $\mathsf{S}_{\mathcal{T}}^{q} \cong [p]$.

Of course, $\mathbf{D}(X)$ cannot be a fractional Calabi–Yau category with a non-integer Calabi–Yau dimension. However, we will see soon some natural and geometrically meaningful fractional Calabi–Yau categories.

1.7 Hochschild Homology and Cohomology

Hochschild homology and cohomology of algebras are important classical invariants. They were also extended to differential graded (DG) algebras and categories [Kel06] and to derived categories of coherent sheaves [Ma01, Ma09]. Basing on these results, one can also define them (under some technical assumptions) for triangulated categories. A non-rigorous definition is the following

$$\mathsf{HH}^{\bullet}(\mathcal{T}) = \mathsf{Ext}^{\bullet}(\mathsf{id}_{\mathcal{T}}, \mathsf{id}_{\mathcal{T}}), \qquad \mathsf{HH}_{\bullet}(\mathcal{T}) = \mathsf{Ext}^{\bullet}(\mathsf{id}_{\mathcal{T}}, \mathsf{S}_{\mathcal{T}}).$$

Thus Hochschild cohomology $\mathsf{HH}^{\bullet}(\mathcal{T})$ is the self-Ext-algebra of the identity functor of $\mathcal{T}$, and Hochschild homology $\mathsf{HH}_{\bullet}(\mathcal{T})$ is the Ext-space from the identity functor to the Serre functor. For a more rigorous definition one should choose a DG-enhancement of the triangulated category $\mathcal{T}$, replace the identity and the Serre functor by appropriate DG-bimodules, and compute Ext-spaces in the derived category of DG bimodules. In a geometrical situation (i.e., when $\mathcal{T} = \mathbf{D}(X)$ with X smooth and projective) one can replace the identity functor by the structure sheaf of the diagonal and the Serre functor by the diagonal pushforward of $\omega_X[\dim X]$ (i.e. Fourier–Mukai kernel of the Serre functor) and compute Ext-spaces in the derived category of coherent sheaves on the square $X \times X$ of the variety:

$$\mathsf{HH}^\bullet(\mathbf{D}(X)) = \mathsf{Ext}^\bullet(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X) \qquad \mathsf{HH}_\bullet(\mathbf{D}(X)) = \mathsf{Ext}^\bullet(\delta_*\mathcal{O}_X, \delta_*\omega_X[\dim X]).$$

In this case, Hochschild homology and cohomology have an interpretation in terms of standard geometrical invariants.

Theorem 1.16 (Hochschild–Kostant–Rosenberg) *If $\mathcal{T} = \mathbf{D}(X)$ with X smooth and projective, then*

$$\mathsf{HH}^k(\mathcal{T}) = \bigoplus_{q+p=k} H^q(X, \Lambda^p T_X), \qquad \mathsf{HH}_k(\mathcal{T}) = \bigoplus_{q-p=k} H^q(X, \Omega_X^p) = \bigoplus_{q-p=k} H^{p,q}(X).$$

Thus the Hochschild cohomology is the cohomology of polyvector fields, while the Hochschild homology is the cohomology of differential forms, or equivalently, the Hodge cohomology of X with one grading lost.

Example 1.17 If $\mathcal{T} = \mathbf{D}(\mathsf{Spec}(\mathsf{k}))$ then $\mathsf{HH}^\bullet(\mathcal{T}) = \mathsf{HH}_\bullet(\mathcal{T}) = \mathsf{k}$. If $\mathcal{T} = \mathbf{D}(C)$ with C being a smooth projective curve of genus g, then

$$\mathsf{HH}^\bullet(\mathcal{T}) = \begin{cases} \mathsf{k} \oplus \mathfrak{sl}_2[-1], & \text{if } g = 0, \\ \mathsf{k} \oplus \mathsf{k}^2[-1] \oplus \mathsf{k}[-2], & \text{if } g = 1, \\ \mathsf{k} \oplus \mathsf{k}^g[-1] \oplus \mathsf{k}^{3g-3}[-2], & \text{if } g \geq 2, \end{cases}$$

and furthermore, for all g one has

$$\mathsf{HH}_\bullet(\mathcal{T}) = \mathsf{k}^g[1] \oplus \mathsf{k}^2 \oplus \mathsf{k}^g[-1].$$

Hochschild cohomology is a graded algebra (in fact, it is what is called a Gerstenhaber algebra, having both an associative multiplication and a Lie bracket of degree -1), and Hochschild homology is a graded module over it. When the category $\mathcal{T}$ is a Calabi–Yau category, it is a free module.

Lemma 1.18 ([K15b]) *If $\mathcal{T}$ is a Calabi–Yau category of dimension $n \in \mathbb{Z}$ then*

$$\mathsf{HH}_\bullet(\mathcal{T}) \cong \mathsf{HH}^\bullet(\mathcal{T})[n].$$

Proof By definition of a Calabi–Yau category we have $\mathsf{S}_\mathcal{T} = \mathsf{id}_\mathcal{T}[n]$. Substituting this into the definition of Hochschild homology we get the result. □

For a more accurate proof one should do the same with DG bimodules or with sheaves on $X \times X$, see [K15b] for details.

2 Semiorthogonal Decompositions

A semiorthogonal decomposition is a way to split a triangulated category into smaller pieces.

2.1 Two-Step Decompositions

We start with the following simplified notion.

Definition 2.1 A (two-step) semiorthogonal decomposition (s.o.d. for short) of a triangulated category $\mathcal{T}$ is a pair of full triangulated subcategories $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ such that

- $\mathsf{Hom}(B, A) = 0$ for any $A \in \mathcal{A}$, $B \in \mathcal{B}$;
- for any $T \in \mathcal{T}$ there is a distinguished triangle

$$T_{\mathcal{B}} \to T \to T_{\mathcal{A}} \to T_{\mathcal{B}}[1] \tag{6}$$

with $T_{\mathcal{A}} \in \mathcal{A}$ and $T_{\mathcal{B}} \in \mathcal{B}$.

An s.o.d. is denoted by $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. Before giving an example let us make some observations.

Lemma 2.2 *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is an s.o.d, then for any $T \in \mathcal{T}$ the triangle (6) is unique up to isomorphism and functorial in T. In particular, the association $T \mapsto T_{\mathcal{A}}$ is a functor $\mathcal{T} \to \mathcal{A}$, left adjoint to the embedding functor $\mathcal{A} \to \mathcal{T}$, and the association $T \mapsto T_{\mathcal{B}}$ is a functor $\mathcal{T} \to \mathcal{B}$, right adjoint to the embedding functor $\mathcal{B} \to \mathcal{T}$.*

Proof Let $T, T' \in \mathcal{T}$ and $\varphi \in \mathsf{Hom}(T, T')$. Let (6) and

$$T'_{\mathcal{B}} \to T' \to T'_{\mathcal{A}} \to T'_{\mathcal{B}}[1]$$

be the corresponding decomposition triangles. Consider the long exact sequence obtained by applying the functor $\mathsf{Hom}(-, T'_{\mathcal{A}})$ to (6):

$$\cdots \to \mathsf{Hom}(T_{\mathcal{B}}[1], T'_{\mathcal{A}}) \to \mathsf{Hom}(T_{\mathcal{A}}, T'_{\mathcal{A}}) \to \mathsf{Hom}(T, T'_{\mathcal{A}}) \to \mathsf{Hom}(T_{\mathcal{B}}, T'_{\mathcal{A}}) \to \dots$$

By semiorthogonality the left and the right terms are zero (recall that $\mathcal{B}$ is triangulated, so $T_{\mathcal{B}}[1] \in \mathcal{B}$). Hence $\mathsf{Hom}(T_{\mathcal{A}}, T'_{\mathcal{A}}) \cong \mathsf{Hom}(T, T'_{\mathcal{A}})$. This means that the composition $T \xrightarrow{\varphi} T' \to T'_{\mathcal{A}}$ factors in a unique way as a composition $T \to T_{\mathcal{A}} \to T'_{\mathcal{A}}$. Denoting the obtained morphism $T_{\mathcal{A}} \to T'_{\mathcal{A}}$ by $\varphi_{\mathcal{A}}$, the uniqueness implies the functoriality of $T \mapsto T_{\mathcal{A}}$. Furthermore, taking an arbitrary object $A \in \mathcal{A}$ and applying the functor $\mathsf{Hom}(-, A)$ to (6) and using again the semiorthogonality, we deduce an isomorphism $\mathsf{Hom}(T_{\mathcal{A}}, A) \cong \mathsf{Hom}(T, A)$, which means that the functor $T \mapsto T_{\mathcal{A}}$ is left adjoint to the embedding $\mathcal{A} \to \mathcal{T}$. The functoriality of $T \mapsto T_{\mathcal{B}}$ and its adjunction are proved analogously. □

Note also that the composition of the embedding $\mathcal{A} \to \mathcal{T}$ with the projection $\mathcal{T} \to \mathcal{A}$ is isomorphic to the identity.

In fact, the above construction can be reversed.

Lemma 2.3 *Assume $\alpha : \mathcal{A} \to \mathcal{T}$ is a triangulated functor, which has a left adjoint $\alpha^* : \mathcal{T} \to \mathcal{A}$ and $\alpha^* \circ \alpha \cong \mathsf{id}_{\mathcal{A}}$ (such $\mathcal{A}$ is called* left admissible*). Then α is full and faithful and there is an s.o.d.*

$$\mathcal{T} = \langle \alpha(\mathcal{A}), \mathsf{Ker}\, \alpha^* \rangle.$$

Analogously, if $\beta : \mathcal{B} \to \mathcal{T}$ is a triangulated functor, which has a right adjoint $\beta^! : \mathcal{T} \to \mathcal{B}$ and $\beta^! \circ \beta \cong \mathsf{id}_{\mathcal{B}}$ (such $\mathcal{B}$ is called right admissible*), then β is full and faithful and there is an s.o.d.*

$$\mathcal{T} = \langle \mathsf{Ker}\, \beta^!, \beta(\mathcal{B}) \rangle.$$

Proof If $B \in \mathsf{Ker}\, \alpha^*$ then by adjunction $\mathsf{Hom}(B, \alpha(A)) = \mathsf{Hom}(\alpha^*(B), A) = 0$. Further, for any $T \in \mathcal{T}$ consider the unit of adjunction $T \to \alpha\alpha^*(T)$ and extend it to a distinguished triangle

$$T' \to T \to \alpha\alpha^*(T) \to T'[1].$$

Applying α^* we get a distinguished triangle

$$\alpha^* T' \to \alpha^* T \to \alpha^*\alpha\alpha^*(T) \to \alpha^* T'[1].$$

If we show that the middle map is an isomorphism, it would follow that $\alpha^* T' = 0$, hence $T' \in \mathsf{Ker}\, \alpha^*$. For this consider the composition $\alpha^* T \to \alpha^*\alpha\alpha^*(T) \to \alpha^* T$, where the first map is the morphism from the triangle (it is induced by the unit of the adjunction), and the second map is induced by the counit of the adjunction. The composition of these maps is an isomorphism by one of the definitions of adjunction. Moreover, the second morphism is an isomorphism by the condition of the Lemma. Hence the first morphism is also an isomorphism. As we noted above, this implies that $T' \in \mathsf{Ker}\, \alpha^*$, and so the above triangle is a decomposition triangle for T. This proves the first semiorthogonal decomposition. The second statement is proved analogously. □

Example 2.4 Let X be a k-scheme with the structure morphism $\pi_X : X \to \mathsf{Spec}(\mathsf{k})$. If $H^\bullet(X, \mathcal{O}_X) = \mathsf{k}$ then there is a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle \mathsf{Ker}\, R\pi_{X*}, L\pi_X^*(\mathbf{D}(\mathsf{Spec}(\mathsf{k}))) \rangle.$$

Indeed, the functor $R\pi_{X*}$ is right adjoint to $L\pi_X^*$ and by projection formula

$$R\pi_{X*}(L\pi_X^*(F)) \cong F \overset{\mathbb{L}}{\otimes} R\pi_{X*}(\mathcal{O}_X) = F \overset{\mathbb{L}}{\otimes} H^\bullet(X, \mathcal{O}_X) = F \overset{\mathbb{L}}{\otimes} \mathsf{k} = F.$$

Therefore, the functor $L\pi_X^*$ is fully faithful and right admissible, so the second semiorthogonal decomposition of Lemma 2.3 applies.

Exercise 2.5 Show that there is also a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle L\pi_X^*(\mathbf{D}(\mathsf{Spec}(\mathsf{k})), \mathsf{Ker}(R\pi_{X*} \circ \mathsf{S}_X)\rangle.$$

Note that $\mathbf{D}(\mathsf{Spec}(\mathsf{k}))$ is the derived category of k-vector spaces (so we will denote it simply by $\mathbf{D}(\mathsf{k})$), its objects can be thought of just as graded vector spaces, and the functor $L\pi_X^*$ applied to a graded vector space $V^\bullet$ is just $V^\bullet \otimes \mathcal{O}_X$ (a complex of trivial vector bundles with zero differentials). Moreover, we have $R\pi_{X*}(-) \cong \mathsf{Ext}^\bullet(\mathcal{O}_X, -)$, so its kernel is the orthogonal subcategory

$$\mathcal{O}_X^\perp = \{F \in \mathbf{D}(X) \mid \mathsf{Ext}^\bullet(\mathcal{O}_X, F) = 0\}.$$

The decompositions of Example 2.4 and Exercise 2.5 thus can be rewritten as

$$\mathbf{D}(X) = \langle \mathcal{O}_X^\perp, \mathcal{O}_X\rangle \quad \text{and} \quad \mathbf{D}(X) = \langle \mathcal{O}_X, {}^\perp\mathcal{O}_X\rangle,$$

where we write just $\mathcal{O}_X$ instead of $\mathcal{O}_X \otimes \mathbf{D}(\mathsf{k})$.

Example 2.6 Let E be an object in $\mathbf{D}(X)$ such that $\mathsf{Ext}^\bullet(E, E) = \mathsf{k}$ (so-called exceptional object). Then there are semiorthogonal decompositions

$$\mathbf{D}(X) = \langle E^\perp, E\rangle \quad \text{and} \quad \mathbf{D}(X) = \langle E, {}^\perp E\rangle,$$

where again we write E instead of $E \otimes \mathbf{D}(\mathsf{k})$, and $E^\perp$, ${}^\perp E$ are defined as

$$E^\perp = \{F \in \mathbf{D}(X) \mid \mathsf{Ext}^\bullet(E, F) = 0\}, \qquad {}^\perp E = \{F \in \mathbf{D}(X) \mid \mathsf{Ext}^\bullet(F, E) = 0\}.$$

2.2 General Semiorthogonal Decompositions

The construction of Lemma 2.3 can be iterated to produce a longer (multi-step) semiorthogonal decomposition.

Definition 2.7 A semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a sequence $\mathcal{A}_1, \dots, \mathcal{A}_m$ of its full triangulated subcategories, such that

(i) $\mathsf{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$, and
(ii) for any $T \in \mathcal{T}$ there is a chain of morphisms

$$0 = T_m \to T_{m-1} \to \cdots \to T_1 \to T_0 = T$$

such that $\mathsf{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$ for each $1 \le i \le m$.

The notation for a semiorthogonal decomposition is $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m \rangle$.

Lemma 2.8 *Assume $E_1, \dots, E_n$ is a sequence of exceptional objects such that $\mathsf{Ext}^\bullet(E_i, E_j) = 0$ for $i > j$ (this is called an* exceptional collection*). Then*

$$\mathbf{D}(X) = \langle E_1^\perp \cap \cdots \cap E_n^\perp, E_1, \dots, E_n \rangle$$

is a semiorthogonal decomposition.

Proof Write the s.o.d. of Example 2.6 for $E_n \in \mathbf{D}(X)$. Then note that $E_1, \dots, E_{n-1}$ are in $E_n^\perp$. Then write analogous s.o.d for the exceptional object E_{n-1} in the triangulated category $E_n^\perp$ and repeat until the required s.o.d. is obtained. □

Remark 2.9 If the first component $E_1^\perp \cap \cdots \cap E_n^\perp$ of the above decomposition is zero, one says that $E_1, \dots, E_n$ is a full exceptional collection.

Remark 2.10 Whenever a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is given, one can construct many other semiorthogonal decompositions by operations called "mutations". Roughly speaking, to mutate a semiorthogonal decomposition we omit one of its components $\mathcal{A}_i$ and then insert a new component in a different position. The new component is abstractly equivalent to $\mathcal{A}_i$, but is embedded differently into $\mathcal{T}$. Mutations induce an action of the braid group on the set of all semiorthogonal decompositions of a given triangulated category.

2.3 Semiorthogonal Decompositions for Fano Varieties

Most interesting examples of semiorthogonal decompositions come from Fano varieties. Recall that a Fano variety is a smooth projective connected variety X with ample anticanonical class $-K_X$. A Fano variety X is prime if $\mathsf{Pic}(X) \cong \mathbb{Z}$. The index of a prime Fano variety X is the maximal integer r such that $-K_X = rH$ for some $H \in \mathsf{Pic}(X)$.

Example 2.11 Let X be a Fano variety of index r with $-K_X = rH$. Then the collection of line bundles $(\mathcal{O}_X((1-r)H), \dots, \mathcal{O}_X(-H), \mathcal{O}_X)$ is an exceptional collection. Indeed, for $i > -r$ we have

$$H^{>0}(X, \mathcal{O}_X(iH)) = H^{>0}(X, K_X((i+r)H)) = 0$$

by Kodaira vanishing. Moreover, $H^0(X, \mathcal{O}_X(iH)) = 0$ for $i < 0$ by ampleness of H and $H^0(X, \mathcal{O}_X) = \mathsf{k}$ by connectedness of X. Applying Lemma 2.8 we obtain a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X((1-r)H), \dots, \mathcal{O}_X(-H), \mathcal{O}_X \rangle$$

with $\mathcal{A}_X$ being the orthogonal complement of the sequence.

In some cases, the orthogonal complement $\mathcal{A}_X$ appearing in the above semiorthogonal decomposition vanishes or can be explicitly described.

Example 2.12 The projective space $\mathbb{P}^n$ is a Fano variety of index $r = n + 1$. The orthogonal complement of the maximal exceptional sequence of line bundles vanishes and we have a full exceptional collection

$$\mathbf{D}(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle.$$

Example 2.13 A smooth quadric $Q^n \subset \mathbb{P}^{n+1}$ is a Fano variety of index $r = n$. The orthogonal complement of the maximal exceptional sequence of line bundles does not vanish but can be explicitly described. In fact, there is a semiorthogonal decomposition

$$\mathbf{D}(Q^n) = \langle \mathbf{D}(\mathsf{Cliff}_0), \mathcal{O}(1-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle,$$

where Cliff_0 is the even part of the corresponding Clifford algebra. If the field k is algebraically closed of characteristic 0 then Cliff_0 is Morita equivalent to k, if n is odd, or to $\mathsf{k} \times \mathsf{k}$, if n is even. So in this case the category $\mathbf{D}(\mathsf{Cliff}_0)$ is generated by one or two (completely orthogonal) exceptional objects, which in terms of $\mathbf{D}(Q^n)$ are given by the spinor bundles.

In fact, the semiorthogonal decomposition of Example 2.13 is much more general. It is valid for arbitrary base fields (of odd characteristic) and also for non-smooth quadrics. There is also an analog for flat families of quadrics over nontrivial base schemes (see [K08]).

If X is a Fano variety of dimension 2 (i.e. a del Pezzo surface) and the base field is $\mathbb{C}$, then $\mathbf{D}(X)$ has a full exceptional collection. This follows easily from representation of X as a blow up of $\mathbb{P}^2$ in several points and Orlov's blowup formula (see Theorem 3.4 below). For more general fields the situation is more complicated.

2.4 Fano Threefolds

The situation becomes much more interesting when one goes into dimension 3. Fano threefolds were completely classified in works of Fano, Iskovskih, Mori and Mukai. There are 105 deformation families of those (quite a large number!), so we restrict our attention to prime Fano threefolds. These form only 17 families.

The index of Fano threefolds is bounded by 4. The only Fano threefold of index 4 is the projective space $\mathbb{P}^3$, and the only Fano threefold of index 3 is the quadric Q^3, their derived categories are described in Examples 2.12 and 2.13. The structure of derived categories of prime Fano threefolds of index 1 and 2 is summarized in Table 1 (see [O91, BO95, K96, K04, K05, K06a, K09a, K14, K15b] for details).

Here $V^{w_0,\dots,w_{r+3}}_{d_1,\dots,d_r}$ denotes a smooth complete intersection of multidegree $(d_1, \dots, d_r)$ in a weighted projective space with weights $(w_0, \dots, w_{r+3})$ (and if

Table 1 Derived categories of prime Fano threefolds

Index 1	Index 2	Relation	Serre functor	$\dim^{CY}$
$\mathbf{D}(X_{22}) = \langle E_4, E_3, E_2, \mathcal{O}\rangle$	$\mathbf{D}(X_5) = \langle E_2(-H), \mathcal{O}(-H), E_2, \mathcal{O}\rangle$			
$\mathbf{D}(X_{18}) = \langle \mathbf{D}(C_2), E_2, \mathcal{O}\rangle$	$\mathbf{D}(V_{2,2}) = \langle \mathbf{D}(C_2), \mathcal{O}(-H), \mathcal{O}\rangle$			
$\mathbf{D}(X_{16}) = \langle \mathbf{D}(C_3), E_3, \mathcal{O}\rangle$				
$\mathbf{D}(X_{14}) = \langle \mathcal{A}_{X_{14}}, E_2, \mathcal{O}\rangle$	$\mathbf{D}(V_3) = \langle \mathcal{A}_{V_3}, \mathcal{O}(-H), \mathcal{O}\rangle$	$\mathcal{A}_{X_{14}} \cong \mathcal{A}_{V_3}$	$\mathsf{S}^3 \cong [5]$	$1\frac{2}{3}$
$\mathbf{D}(X_{12}) = \langle \mathbf{D}(C_7), E_5, \mathcal{O}\rangle$				
$\mathbf{D}(X_{10}) = \langle \mathcal{A}_{X_{10}}, E_2, \mathcal{O}\rangle$	$\mathbf{D}(dS_4) = \langle \mathcal{A}_{dS_4}, \mathcal{O}(-H), \mathcal{O}\rangle$	$\mathcal{A}_{X_{10}} \sim \mathcal{A}_{dS_4}$	$\mathsf{S}^2 \cong [4]$	2
$\mathbf{D}(V_{2,2,2}) = \langle \mathcal{A}_{V_{2,2,2}}, \mathcal{O}\rangle$				
$\mathbf{D}(V_{2,3}) = \langle \mathcal{A}_{V_{2,3}}, E_2, \mathcal{O}\rangle$	$\mathbf{D}(V_6^{1,1,1,2,3}) = \langle \mathcal{A}_{V_6}, \mathcal{O}(-H), \mathcal{O}\rangle$	$\mathcal{A}_{V_{2,3}} \sim \mathcal{A}_{V_6}$	$\mathsf{S}^3 \cong [7]$	$2\frac{1}{3}$
$\mathbf{D}(V_4) = \langle \mathcal{A}_{V_4}, \mathcal{O}\rangle$			$\mathsf{S}^4_{\mathcal{A}_{V_4}} \cong [10]$	$2\frac{1}{2}$
$\mathbf{D}(dS_6) = \langle \mathcal{A}_{dS_6}, \mathcal{O}\rangle$			$\mathsf{S}^6_{\mathcal{A}_{dS_6}} \cong [16]$	$2\frac{2}{3}$

all weights are equal to 1 we omit them). The notation X_d means a prime Fano threefold of degree $d = H^3$. Further, dS_d stands for the degree d double solid, i.e., the double covering of $\mathbb{P}^3$ branched in a smooth divisor of degree d.

As we discussed in Example 2.11 the structure sheaf $\mathcal{O}_X$ on a Fano threefold X is always exceptional, and in case of index 2 (the second column of the table) it extends to an exceptional pair $(\mathcal{O}_X(-H), \mathcal{O}_X)$ (where as usual H denotes the ample generator of the Picard group). Whenever there is an additional exceptional vector bundle of rank r, we denote it by E_r (see one of the above references for the construction of these). For Fano threefolds X_{22} and X_5 (the first line of the table) one can construct in this way a full exceptional collection.

For other prime Fano threefolds there is no full exceptional collection (this follows from Corollary 2.16 below), so the derived category contains an additional component. In four cases (varieties X_{18}, X_{16}, X_{12}, and $V_{2,2}$) this component can be identified with the derived category of a curve (of genus 2, 3, and 7 denoted by C_2, C_3, and C_7 respectively). For the other Fano threefolds, the extra component $\mathcal{A}_X$ cannot be described in a simple way. However, it has some interesting properties. For example, in most cases it is a fractional Calabi–Yau category (as defined in Definition 1.15, see [K15b] for the proofs). We list in the fourth column of the table the CY property of the Serre functor of these categories, and in the fifth column their CY dimension.

It is a funny and mysterious observation that the nontrivial components of Fano threefolds sitting in the same row of the table have very much in common. In fact they are equivalent for rows 1, 2, and 4, and are expected to be deformation equivalent for rows 6 and 8 (see [K09a] for precise results and conjectures). We mention this relation in the third column of the table, by using the sign "≅" for equivalence and the sign "∼" for deformation equivalence of categories.

It is interesting to compare this table with the table in Sect. 2.3 of [Bea15], where the known information about birational properties of prime Fano threefolds is collected (the notation we use agrees with the notation in loc. cit.). From the comparison it is easy to see that the structure of the derived categories of Fano threefolds correlates with their rationality properties. All Fano threefolds which are known to be rational (X_{22}, X_{18}, X_{16}, X_{12}, X_5, and $V_{2,2}$) have semiorthogonal decompositions consisting only of exceptional objects (derived categories of points) and derived categories of curves. On a contrary, those Fano threefolds which are known or expected to be nonrational, have some "nontrivial pieces" in their derived categories. Note also that typically these "nontrivial pieces" are fractional Calabi–Yau categories.

In the next section we will try to develop this observation into a birational invariant.

Remark 2.14 Among the most interesting examples of nonrational varieties are the Artin–Mumford double solids (see [AM72]). If X is one of these double solids and $\widetilde{X}$ is its blowup then there is a semiorthogonal decomposition

$$\mathbf{D}(\widetilde{X}) = \langle \mathcal{A}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(-H), \mathcal{O}_{\widetilde{X}} \rangle.$$

The category $\mathcal{A}_{\widetilde{X}}$ is equivalent to the derived category of the associated Enriques surface (see [IK15, HT15]). This explains the torsion in the cohomology of X which implies its non-rationality.

2.5 *Hochschild Homology and Semiorthogonal Decompositions*

A fundamental property of Hochschild homology, which makes it very useful, is its additivity with respect to semiorthogonal decompositions.

Theorem 2.15 ([K09b]) *If $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is a semiorthogonal decomposition, then*

$$\mathsf{HH}_\bullet(\mathcal{T}) = \bigoplus_{i=1}^{m} \mathsf{HH}_\bullet(\mathcal{A}_i).$$

One of the nice consequences of this result is the following necessary condition for a category to have a full exceptional collection.

Corollary 2.16 *If a category $\mathcal{T}$ has a full exceptional collection then $\mathsf{HH}_k(\mathcal{T}) = 0$ for $k \neq 0$ and $\dim \mathsf{HH}_0(\mathcal{T}) < \infty$. Moreover, the length of any full exceptional collection in $\mathcal{T}$ equals $\dim \mathsf{HH}_0(\mathcal{T})$.*

In particular, if X is a smooth projective variety and $\mathbf{D}(X)$ has a full exceptional collection, then $H^{p,q}(X) = 0$ for $p \neq q$, and the length of the exceptional collection equals to $\sum \dim H^{p,p}(X)$.

Proof The first part follows from the additivity Theorem and Example 1.17. The second part follows from the first and the Hochschild–Kostant–Rosenberg isomorphism of Theorem 1.16. □

Remark 2.17 Note that the conditions of the Corollary are only necessary. The simplest example showing they are no sufficient is provided by the derived category of an Enriques surface S. It is well known that the Hodge numbers of S are zero away from the diagonal of the Hodge diamond, however a full exceptional collection in $\mathbf{D}(S)$ cannot exist since the Grothendieck group $K_0(\mathbf{D}(S))$ has torsion.

In a contrast, the Hochschild cohomology is not additive (it depends not only on the components of a semiorthogonal decomposition, but also on the way they are glued together). However, if there is a completely orthogonal decomposition, then the Hochschild cohomology is additive.

Lemma 2.18 ([K09b]) *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a completely orthogonal decomposition then*

$$\mathsf{HH}^\bullet(\mathcal{T}) = \mathsf{HH}^\bullet(\mathcal{A}) \oplus \mathsf{HH}^\bullet(\mathcal{B}).$$

This result also has a nice consequence. Recall that for $\mathcal{T} = \mathbf{D}(X)$ we have $\mathsf{HH}^0(\mathcal{T}) = H^0(X, \mathcal{O}_X)$ by the HKR isomorphism. This motivates the following

Definition 2.19 A triangulated category $\mathcal{T}$ is called connected, if $\mathsf{HH}^0(\mathcal{T}) = \mathsf{k}$.

Corollary 2.20 *If $\mathcal{T}$ is a connected triangulated category then $\mathcal{T}$ has no completely orthogonal decompositions.*

Proof If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a completely orthogonal decomposition then we have $\mathsf{k} = \mathsf{HH}^0(\mathcal{T}) = \mathsf{HH}^0(\mathcal{A}) \oplus \mathsf{HH}^0(\mathcal{B})$, hence one of the summands vanishes. But a nontrivial category always has nontrivial zero Hochschild cohomology (since the identity functor always has the identity endomorphism). □

2.6 Indecomposability

There are triangulated categories which have no nontrivial semiorthogonal decompositions. We will call such categories indecomposable. First examples of indecomposable categories were found by Bridgeland in [Bri99].

Proposition 2.21 ([K15b]) *If $\mathcal{T}$ is a connected Calabi–Yau category then $\mathcal{T}$ is indecomposable.*

Proof Assume $\mathcal{T}$ is a Calabi–Yau category of dimension n and $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition. For any $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have

$$\mathsf{Hom}(A, B)^\vee = \mathsf{Hom}(B, \mathsf{S}_{\mathcal{T}}(A)) = \mathsf{Hom}(B, A[n]) = 0$$

since $A[n] \in \mathcal{A}$. Hence the decomposition is completely orthogonal and Corollary 2.20 applies. □

Corollary 2.22 *The category $\mathbf{D}(\mathsf{k})$ is indecomposable.*

Proof Indeed, $\mathsf{S}_{\mathbf{D}(\mathsf{k})} = \mathsf{id}$, so it is Calabi–Yau of dimension 0. $\square$

Besides, one can check that derived categories of curves of positive genus are indecomposable.

Proposition 2.23 ([Oka11]) *Let $\mathcal{T} = \mathbf{D}(C)$ with C a smooth projective curve. If $g(C) > 0$ then $\mathcal{T}$ is indecomposable. If $C = \mathbb{P}^1$ then any semiorthogonal decomposition of $\mathcal{T}$ is given by an exceptional pair.*

One can also show that many surfaces have no semiorthogonal decompositions, see [KO12].

The Bridgeland's result shows that Calabi–Yau categories can be thought of as the simplest building blocks of other categories. Because of that the following observation is useful.

Lemma 2.24 ([K15b]) *Assume $\mathbf{D}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition with $\mathcal{A}$ being Calabi–Yau category of dimension n. Then $\dim X \geq n$.*

Proof By Calabi–Yau property of $\mathcal{A}$ we have $\mathsf{HH}_{-n}(\mathcal{A}) = \mathsf{HH}^0(\mathcal{A}) \neq 0$ (see Lemma 1.18), and by additivity of Hochschild homology $\mathsf{HH}_{-n}(\mathbf{D}(X)) \neq 0$ as well. But by HKR isomorphism if $n > \dim X$ then $\mathsf{HH}_{-n}(\mathbf{D}(X)) = 0$. $\square$

In fact, most probably in the boundary case $\dim X = n$ there are also strong restrictions.

Conjecture 2.25 ([K15b]) Assume $\mathbf{D}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition with $\mathcal{A}$ being Calabi–Yau category of dimension $n = \dim X$. Then X is a blowup of a Calabi–Yau variety Y of dimension n and $\mathcal{A} \cong \mathbf{D}(Y)$.

We will see in the next section that the converse is true.

Unfortunately, analogs of these results for fractional Calabi–Yau categories are not known, and in fact, some of them are just not true. For example, if X is a smooth cubic surface in $\mathbb{P}^3$ and $\mathcal{A}_X = \mathcal{O}_X^\perp \subset \mathbf{D}(X)$, then $\mathcal{A}_X$ is a fractional Calabi–Yau category of dimension $4/3$, but it is decomposable (in fact, it has a full exceptional collection).

3 Griffiths Components

In this section we discuss the behavior of derived categories under standard birational transformations. We start with a relative version of splitting off an exceptional object.

3.1 Relative Exceptional Objects

Assume $f : X \to Y$ is a morphism of smooth projective varieties. Let $\mathcal{E} \in \mathbf{D}(X)$ be an object such that

$$Rf_* \,\mathsf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_Y. \tag{7}$$

When $Y = \mathsf{Spec}(\mathsf{k})$, the above condition is just the definition of an exceptional object.

Lemma 3.1 *If $\mathcal{E}$ enjoys (7) then the functor $\mathbf{D}(Y) \to \mathbf{D}(X)$, $F \mapsto \mathcal{E} \overset{\mathbb{L}}{\otimes} Lf^*(F)$ is fully faithful and gives a semiorthogonal decomposition*

$$\mathbf{D}(X) = \langle \mathsf{Ker}\, Rf_* \,\mathsf{R}\mathcal{H}om(\mathcal{E}, -), \mathcal{E} \overset{\mathbb{L}}{\otimes} Lf^*(\mathbf{D}(Y))\rangle.$$

Proof Indeed, $Rf_* \,\mathsf{R}\mathcal{H}om(\mathcal{E}, -)$ is the right adjoint of $\mathcal{E} \overset{\mathbb{L}}{\otimes} Lf^*(-)$, and since

$$Rf_* \,\mathsf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E} \overset{\mathbb{L}}{\otimes} Lf^*(F)) \cong Rf_*(\mathsf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}) \overset{\mathbb{L}}{\otimes} Lf^*(F)) \cong Rf_* \,\mathsf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}) \overset{\mathbb{L}}{\otimes} F,$$

the condition (7) and Lemma 2.3 prove the result. □

Example 3.2 Let $X = \mathbb{P}_Y(\mathcal{V}) \xrightarrow{f} Y$ be the projectivization of a vector bundle $\mathcal{V}$ of rank r on Y. Then $Rf_*\mathcal{O}_X \cong \mathcal{O}_Y$, hence any line bundle on X satisfies (7). So, iterating the construction of Lemma 3.1 (along the lines of Example 2.12) we get Orlov's s.o.d. for the projectivization

$$\mathbf{D}(\mathbb{P}_Y(\mathcal{V})) = \langle \mathcal{O}(1-r) \otimes Lf^*(\mathbf{D}(Y)), \dots, \mathcal{O}(-1) \otimes Lf^*(\mathbf{D}(Y)), Lf^*(\mathbf{D}(Y))\rangle. \tag{8}$$

Example 3.3 Let $\mathcal{Q} \subset \mathbb{P}_Y(\mathcal{V})$ be a flat family of quadrics with $f : \mathcal{Q} \to Y$ being the projection. Then similarly we get a semiorthogonal decomposition

$$\mathbf{D}(\mathcal{Q}) = \langle \mathbf{D}(Y, \mathsf{Cliff}_0), \mathcal{O}(3-r) \otimes Lf^*(\mathbf{D}(Y)), \dots, \mathcal{O}(-1) \otimes Lf^*(\mathbf{D}(Y)), Lf^*(\mathbf{D}(Y))\rangle, \tag{9}$$

where Cliff_0 is the sheaf of even parts of Clifford algebras on Y associated with the family of quadrics $\mathcal{Q}$ (see [K08] for details).

3.2 Semiorthogonal Decomposition of a Blowup

The most important for the birational geometry is the following semiorthogonal decomposition Let $X = \mathsf{Bl}_Z(Y)$ be the blowup of a scheme Y in a locally complete intersection subscheme $Z \subset Y$ of codimension c. Then we have the following blowup diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & E \arrow[l, "i"'] \arrow[d, "p"] \arrow[r, equal] & \mathbb{P}_Z(\mathcal{N}_{Z/Y}) \\ Y & Z \arrow[l, "j"'] \end{tikzcd}$$

where the exceptional divisor E is isomorphic to the projectivization of the normal bundle, and its natural map to Z is the standard projection of the projectivization. Note also that under this identification, the normal bundle $\mathcal{O}_E(E)$ of the exceptional divisor is isomorphic to the Grothendieck line bundle $\mathcal{O}_E(-1)$

$$\mathcal{O}_E(E) \cong \mathcal{O}_E(-1) \tag{10}$$

on the projectivization $E \cong \mathbb{P}_Z(\mathcal{N}_{Z/Y})$. We will use the powers of this line bundle to construct functors from $\mathbf{D}(Z)$ to $\mathbf{D}(X)$. For each $k \in \mathbb{Z}$ we consider the Fourier–Mukai functor with kernel $\mathcal{O}_E(k)$, i.e.

$$\Phi_{\mathcal{O}_E(k)}(F) = Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(-)) \colon \mathbf{D}(Z) \to \mathbf{D}(X). \tag{11}$$

Theorem 3.4 (Orlov's Blowup Formula) *The functor $\Phi_{\mathcal{O}_E(k)} : \mathbf{D}(Z) \to \mathbf{D}(X)$ is fully faithful for each k as well as the functor $Lf^* : \mathbf{D}(Y) \to \mathbf{D}(X)$. Moreover, they give the following semiorthogonal decomposition*

$$\mathbf{D}(X) = \langle \Phi_{\mathcal{O}_E(1-c)}(\mathbf{D}(Z)), \dots, \Phi_{\mathcal{O}_E(-1)}(\mathbf{D}(Z)), Lf^*(\mathbf{D}(Y)) \rangle. \tag{12}$$

Sketch of Proof First, $Rf_*\mathcal{O}_X \cong \mathcal{O}_Y$ by a local computation (locally we can assume that the ideal of Z is generated by c functions, this allows to embed X explicitly into $Y \times \mathbb{P}^{c-1}$ and to write down an explicit resolution for its structure sheaf; pushing it forward to Y proves the claim). Hence by Lemma 3.1 the functor Lf^* is fully faithful.

Further, the right adjoint functor $\Phi^!_{\mathcal{O}_E(k)}$ of $\Phi_{\mathcal{O}_E(k)}$ is given by

$$\begin{aligned} F \mapsto Rp_*(\mathcal{O}_E(-k) \overset{\mathbb{L}}{\otimes} i^!(F)) &= Rp_*(\mathcal{O}_E(-k) \overset{\mathbb{L}}{\otimes} Li^*(F) \otimes \mathcal{O}_E(E)[-1]) \\ &= Rp_*(\mathcal{O}_E(-k-1) \overset{\mathbb{L}}{\otimes} Li^*(F)[-1]), \end{aligned}$$

and hence the composition $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(k)}$ is given by

$$\begin{aligned} F &\mapsto Rp_*(\mathcal{O}_E(-k-1) \overset{\mathbb{L}}{\otimes} Li^*(Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F))))[-1] \\ &\cong Rp_*(\mathcal{O}_E(-1) \overset{\mathbb{L}}{\otimes} Li^*(Ri_*(Lp^*(F))))[-1]. \end{aligned}$$

Note that i is a divisorial embedding, hence the composition $Li^* \circ Ri_*$ comes with a distinguished triangle

$$G \otimes \mathcal{O}_E(1)[1] \to Li^* Ri_*(G) \to G \to G \otimes \mathcal{O}_E(1)[2]$$

(this is a derived category version of the "fundamental local isomorphism" $\mathsf{Tor}_p(i_*G, i_*\mathcal{O}_E) \cong G \otimes \Lambda^p \mathcal{N}^\vee_{E/X}$ together with $\mathcal{N}^\vee_{E/X} \cong \mathcal{O}_E(-E) \cong \mathcal{O}_E(1)$, see [H06, Corollary 11.4(ii)]). Therefore we have a distinguished triangle

$$Rp_*(Lp^*(F)) \to \Phi^!_{\mathcal{O}_E(k)} \Phi_{\mathcal{O}_E(k)}(F) \to Rp_*(\mathcal{O}_E(-1) \overset{\mathbb{L}}{\otimes} Lp^*(F))[-1] \to Rp_*(Lp^*(F))[1].$$

Using the projection formula and the fact that $Rp_*\mathcal{O}_E \cong \mathcal{O}_Z$, $Rp_*\mathcal{O}_E(-1) = 0$, we conclude that $\Phi^!_{\mathcal{O}_E(k)} \Phi_{\mathcal{O}_E(k)}(F) \cong F$, hence $\Phi_{\mathcal{O}_E(k)}$ is fully faithful.

Analogously, computing the composition $\Phi^!_{\mathcal{O}_E(k)} \Phi_{\mathcal{O}_E(l)}$ we get a distinguished triangle

$$Rp_*(\mathcal{O}_E(l-k) \overset{\mathbb{L}}{\otimes} Lp^*(F)) \to \Phi^!_{\mathcal{O}_E(k)} \Phi_{\mathcal{O}_E(l)}(F) \to Rp_*(\mathcal{O}_E(l-k-1) \overset{\mathbb{L}}{\otimes} Lp^*(F))[-1] \to \dots$$

Recall that $Rp_*\mathcal{O}_E(-t) = 0$ for $1 \le t \le c-1$. As for $1-c \le l < k \le -1$ we have $1-c \le l-k$, $l-k-1 \le -1$, hence we have $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(l)} = 0$. This shows that the first $c-1$ components of (12) are semiorthogonal.

Finally, for the composition $Rf_* \circ \Phi_{\mathcal{O}_E(k)}$ we have

$$\begin{aligned} Rf_* \circ \Phi_{\mathcal{O}_E(k)}(F) &= Rf_* Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F)) \\ &\cong Rj_* Rp_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F)) \cong Rj_*(Rp_*\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} F), \end{aligned}$$

and since $Rp_*\mathcal{O}_E(k) = 0$ for $1-c \le k \le -1$, it follows that the composition is zero, and hence the last component of (12) is semiorthogonal to the others.

It remains to show that the components we just described generate the whole category $\mathbf{D}(X)$. This computation is slightly technical. Basically one can compute the composition $Lf^* \circ Rj_*$ and show that it equals $Ri_* \circ Lp^*$ modulo the first $c-1$ components of (12). This means that the RHS of (12) contains the subcategory $\Phi_{\mathcal{O}_E}(\mathbf{D}(Z))$. By (8), it then follows that $Ri_*(\mathbf{D}(E))$ is contained in the RHS, hence any object in the orthogonal is in the kernel of Li^*, hence is supported on $X \setminus E$. But f defines an isomorphism $X \setminus E \cong Y \setminus Z$, hence any such object F can be written as $Lf^*(Rf_*(F))$, hence still lives in the RHS of (12). Altogether, this argument proves the Theorem. □

Roughly speaking, we can interpret Theorem 3.4 by saying that the "difference" between the derived categories of X and Y is given by a number of derived categories of subvarieties of dimension $\le n-2$, where $n = \dim X = \dim Y$. On the other hand, by Weak Factorization Theorem any two birational varieties can be linked by a sequence of blowups and blowdowns with smooth (and hence lci) centers.

Thus one can get the resulting derived category from the original one by an iterated "addition" and "subtraction" of derived categories of smooth projective varieties of dimension at most $n-2$.

3.3 Griffiths Components

The above observation suggests the following series of definitions.

Definition 3.5 Define the geometric dimension of a triangulated category $\mathcal{A}$, $\mathrm{gdim}(\mathcal{A})$, as the minimal integer k such that $\mathcal{A}$ can be realized as an admissible subcategory of a smooth projective and connected variety of dimension k.

Example 3.6 If $\mathcal{A}$ is a triangulated category of geometric dimension 0 then $\mathcal{A} \cong \mathbf{D}(\mathsf{k})$. Indeed, by definition $\mathcal{A}$ should be an admissible subcategory of the derived category of a smooth projective connected variety of dimension 0, i.e., of $\mathbf{D}(\mathsf{Spec}(\mathsf{k})) = \mathbf{D}(\mathsf{k})$. But this category is indecomposable (see Corollary 2.22), hence $\mathcal{A} \cong \mathbf{D}(\mathsf{k})$.

Example 3.7 If $\mathcal{A}$ is an indecomposable triangulated category of geometric dimension 1 then $\mathcal{A} \cong \mathbf{D}(C)$, where C is a curve of genus $g \geq 1$. Indeed, by definition $\mathcal{A}$ should be an admissible subcategory of the derived category of a smooth projective connected variety of dimension 1, i.e., of $\mathbf{D}(C)$. If $g(C) \geq 1$ then $\mathbf{D}(C)$ is indecomposable by Proposition 2.23 and so $\mathcal{A} = \mathbf{D}(C)$. If $g(C) = 0$, i.e., $C \cong \mathbb{P}^1$, then again by Proposition 2.23 any nontrivial decomposition of $\mathbf{D}(C)$ consists of two exceptional objects, so if $\mathcal{A} \subset \mathbf{D}(C)$ is an indecomposable admissible subcategory then $\mathcal{A} \cong \mathbf{D}(\mathsf{k})$, but then its geometric dimension is 0.

A classification of triangulated categories of higher geometrical dimension (if possible) should be much more complicated. For instance, it was found out recently that some surfaces of general type with $p_g = q = 0$ contain admissible subcategories with zero Hochschild homology (so called quasiphantom categories). These categories are highly nontrivial examples of categories of geometric dimension 2.

Definition 3.8 A semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ is maximal, if each component $\mathcal{A}_i$ is an indecomposable category, i.e., does not admit a nontrivial semiorthogonal decomposition.

Definition 3.9 Let $\mathbf{D}(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ be a maximal semiorthogonal decomposition. Let us say that its component $\mathcal{A}_i$ is a Griffiths component, if

$$\mathrm{gdim}(\mathcal{A}_i) \geq \dim X - 1.$$

We denote by

$$\mathsf{Griff}(X) := \{\mathcal{A}_i \mid \mathrm{gdim}(\mathcal{A}_i) \geq \dim X - 1\}$$

the set of Griffiths components of a given maximal semiorthogonal decomposition of $\mathbf{D}(X)$.

One could hope that the set of Griffiths components $\mathsf{Griff}(X)$ is well defined (does not depend on the choice of a maximal semiorthogonal decomposition). Let us imagine this is the case and try to discuss rationality of Fano threefolds on this basis. First, note the following

Lemma 3.10 *If the set* $\mathsf{Griff}(X)$ *of Griffiths components is well defined then it is a birational invariant.*

Proof Recall that by Weak Factorization Theorem if X and Y are smooth projective varieties birational to each other then there is a sequence of blowups and blowdowns with smooth centers connecting X and Y. So, to show that the set of Griffiths components is a birational invariant, it is enough to check that it does not change under a smooth blowup. So, assume that $X = \mathsf{Bl}_Z(Y)$. Let $\mathbf{D}(Y) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ and $\mathbf{D}(Z) = \langle \mathcal{B}_1, \dots, \mathcal{B}_k \rangle$ be maximal semiorthogonal decompositions. Then by Theorem 3.4 we have a semiorthogonal decomposition of $\mathbf{D}(X)$ with components $\mathcal{A}_i$ and $\mathcal{B}_j$ (repeated $c - 1$ times, where c is the codimension of Z). Note also, that $\mathcal{B}_j$ is an admissible subcategory of $\mathbf{D}(Z)$, hence

$$\operatorname{gdim}(\mathcal{B}_j) \le \dim Z \le \dim X - 2,$$

hence none of $\mathcal{B}_j$ is a Griffiths component of $\mathbf{D}(X)$. Thus the sets of Griffiths components of $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ coincide. □

Corollary 3.11 *If the set* $\mathsf{Griff}(X)$ *of Griffiths components is well defined then any rational variety of dimension at least* 2 *has no Griffiths components.*

Proof By previous Lemma it is enough to show that $\mathbb{P}^n$ has no Griffiths components. But by Example 2.12 there is a semiorthogonal decomposition of $\mathbf{D}(\mathbb{P}^n)$ with all components being derived categories of points. Their geometrical dimension is zero, so as soon as $n \ge 2$, none of them is a Griffiths component. □

This "birational invariant" can be effectively used. Consider, for example, a smooth cubic threefold V_3. As it was discussed in Sect. 2.4, it has a semiorthogonal decomposition with two exceptional objects and a category $\mathcal{A}_{V_3}$ as components.

Proposition 3.12 *If the set* $\mathsf{Griff}(X)$ *of Griffiths components is well defined then for a smooth cubic threefold it is nonempty. In particular, a smooth cubic threefold is not rational.*

Proof If the set of Griffiths components of a smooth cubic threefold would be empty, then the category $\mathcal{A} := \mathcal{A}_{V_3}$ should have a semiorthogonal decomposition with components of geometrical dimension at most 1, i.e. by Examples 3.6 and 3.7 with components $\mathbf{D}(\mathsf{k})$ and $\mathbf{D}(C_i)$ with $g(C_i) \ge 1$. To show this is not true we will use Hochschild homology.

Recall that by HKR isomorphism the Hochschild homology of $\mathbf{D}(V_3)$ can be computed in terms of its Hodge numbers. On the other hand, by Griffiths Residue

Theorem the Hodge diamond of V_3 looks as

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

hence $\mathsf{HH}_\bullet(\mathbf{D}(V_3)) = \mathsf{k}^5[1] \oplus \mathsf{k}^4 \oplus \mathsf{k}^5[-1]$. To obtain the category $\mathcal{A}_{V_3}$ we split off two exceptional bundles, hence by additivity of Hochschild homology (Theorem 2.15) we get

$$\mathsf{HH}_\bullet(\mathcal{A}_{V_3}) \cong \mathsf{k}^5[1] \oplus \mathsf{k}^2 \oplus \mathsf{k}^5[-1].$$

Now assume $\mathcal{A}_{V_3}$ has a semiorthogonal decomposition with m components equivalent to $\mathbf{D}(\mathsf{k})$ and k components equivalent to $\mathbf{D}(C_1), \dots, \mathbf{D}(C_k)$. Then by additivity of Hochschild homology and Example 1.17 we get

$$\sum_{i=1}^{k} g(C_i) = \dim \mathsf{HH}_1(\mathcal{A}) = 5, \qquad m + 2k = \dim \mathsf{HH}_0(\mathcal{A}) = 2.$$

It follows from the first that $k \geq 1$ and from the second $k \leq 1$. Hence $k = 1$, $g(C_1) = 5$, and $m = 0$. So, the only possibility is if $\mathcal{A} \cong \mathbf{D}(C)$ with $g(C) = 5$. But $\mathsf{S}_{\mathcal{A}}^3 \cong [5]$, while $\mathsf{S}_{\mathbf{D}(C)}$ clearly does not have this property. □

The same argument works for X_{14}, X_{10}, dS_4, $V_{2,3}$, $V_6^{1,1,1,2,3}$, V_4, and dS_6, i.e. for all prime Fano threefolds with the exception of those which are known to be rational, and $V_{2,2,2}$. For the last one we need another way to check that an equivalence $\mathcal{A}_{V_{2,2,2}} \cong \mathbf{D}(C)$ is impossible. One of the possibilities is by comparing the Hochschild cohomology of these categories.

Remark 3.13 Most probably, the category $\mathcal{A}_{V_3}$ (as well as the other similar categories) is indecomposable. However, this is not so easy to prove, but even without this the above argument works fine.

3.4 Bad News

Unfortunately, Griffiths components are not well defined. To show this we will exhibit a contradiction with Corollary 3.11 by constructing an admissible subcategory of geometric dimension greater than 1 in a rational threefold. The construction is based on the following interesting category discovered by Alexei Bondal in 1990s.

Consider the following quiver with relations

$$Q = \left(\bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bullet \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bullet \;\middle|\; \beta_1\alpha_2 = \beta_2\alpha_1 = 0 \right) \tag{13}$$

As any oriented quiver it has a full exceptional collection

$$\mathbf{D}(Q) = \langle P_1, P_2, P_3 \rangle \tag{14}$$

with P_i being the projective module of the ith vertex. On the other hand, it has an exceptional object

$$E = \left(\mathsf{k} \underset{0}{\overset{1}{\rightrightarrows}} \mathsf{k} \underset{0}{\overset{1}{\rightrightarrows}} \mathsf{k} \right) \tag{15}$$

It gives a semiorthogonal decomposition

$$\mathbf{D}(Q) = \langle \mathcal{A}_0, E \rangle,$$

and it is interesting that the category $\mathcal{A}_0 := E^{\perp}$ has no exceptional objects. Indeed, a straightforward computation shows that the Euler form on $K_0(\mathcal{A}_0)$ is skew-symmetric, and so in $K_0(\mathcal{A}_0)$ there are no vectors of square 1.

In particular, any indecomposable admissible subcategory of $\mathcal{A}_0$ has geometric dimension greater than 1. Indeed, by Example 3.6 and the above argument we know that $\mathcal{A}_0$ has no admissible subcategories of geometric dimension 0. Furthermore, if it would have an admissible subcategory of geometric dimension 1, then by (3.7) this subcategory should be equivalent to $\mathbf{D}(C)$ with $g(C) \geq 1$. But then by additivity of Hochschild homology $\dim \mathsf{HH}_1(\mathcal{A}_0) \geq \dim \mathsf{HH}(\mathbf{D}(C)) \geq g(C) \geq 1$, which contradicts $\mathsf{HH}_1(\mathbf{D}(Q)) = 0$ again by additivity and by (14).

It remains to note that by [K13] there is a rational threefold X with $\mathbf{D}(Q)$ (and a fortiori $\mathbf{D}(\mathcal{A}_0)$) being a semiorthogonal component of $\mathbf{D}(X)$.

Remark 3.14 The same example shows that semiorthogonal decompositions do not satisfy Jordan–Hölder property.

However, as one can see on the example of prime Fano threefolds, there is a clear correlation between appearance of nontrivial pieces in derived categories and nonrationality. Most probably, the notion of a Griffiths component should be modified in some way, and then it will be a birational invariant. One of the possibilities is to include fractional Calabi–Yau property into the definition. Another possibility is to consider minimal categorical resolutions of singular varieties as geometric categories and thus redefine the notion of the geometric dimension. Indeed, as we will see below the category $\mathcal{A}_0$ is of this nature.

3.5 Minimal Categorical Resolution of a Nodal Curve

The category $\mathcal{A}_0$ from the previous section has a nice geometric interpretation. It is a minimal categorical resolution of singularities of a rational nodal curve.

Let $C_0 := \{y^2 z = x^2(x-z)\} \subset \mathbb{P}^2$ be a rational nodal curve and denote by $\mathbf{p}_0 \in C_0$ its node. The normalization map $C \cong \mathbb{P}^1 \xrightarrow{\nu} C_0$ is not a "categorical resolution", since $\nu_* \mathcal{O}_C \ncong \mathcal{O}_{C_0}$ and hence the pullback functor $L\nu^*$ from the derived category of C_0 to the derived category of C is not fully faithful. However, following the recipe of [KL12], one can construct a categorical resolution of C_0 by gluing $\mathbf{D}(C)$ with $\mathbf{D}(\mathbf{p}_0) \cong \mathbf{D}(\mathsf{k})$ along $\nu^{-1}(\mathbf{p}_0) = \{\mathbf{p}_1, \mathbf{p}_2\}$. The resulting category is equivalent to the derived category $\mathbf{D}(Q)$ of Bondal's quiver.

Indeed, the subcategory of $\mathbf{D}(Q)$ generated by the first two vertices (or, equivalently, by the first two projective modules P_1 and P_2) of the quiver is equivalent to $\mathbf{D}(\mathbb{P}^1) \cong \mathbf{D}(C)$, so that the two arrows α_1 and α_2 between the first two vertices correspond to the homogeneous coordinates $(x_1 : x_2)$ on $\mathbb{P}^1$. The third vertex gives $\mathbf{D}(\mathbf{p}_0)$, and the two arrows between the second and the third vertex of Q correspond to the points $\mathbf{p}_1$ and $\mathbf{p}_2$. Finally, the relations of the quiver follow from $x_1(\mathbf{p}_2) = x_2(\mathbf{p}_1) = 0$, the vanishing of (suitably chosen) homogeneous coordinates of $\mathbb{P}^1$ at points $\mathbf{p}_1$ and $\mathbf{p}_2$.

The resolution functor takes an object $F \in \mathbf{D}(C_0)$ to the representation of Q defined by

$$(H^\bullet(C, L\nu^* F \otimes \mathcal{O}_C(-1)), H^\bullet(C, L\nu^* F), Li_0^* F),$$

where $i_0 : \mathbf{p}_0 \to C_0$ is the embedding of the node. It is easy to see that its image is contained in the subcategory $\mathcal{A}_0 = P^\perp \subset \mathbf{D}(Q)$, so the latter can be considered as a categorical resolution of C_0 as well.

If we consider the category $\mathcal{A}_0$ as a geometric category of dimension 1, so that $\mathrm{gdim}(\mathcal{A}_0) = 1$, then $\mathcal{A}_0$ is no longer a Griffiths component of the rational threefold X from Sect. 3.4, and this is no longer a counterexample.

4 Higher Dimensional Varieties

Probably, the most interesting (from the birational point of view) example of a 4-dimensional variety is a cubic fourfold. There are examples of cubic fourfolds which are known to be rational, but general cubic fourfolds are expected to be nonrational.

In this section we will discuss how does rationality of cubic fourfolds correlate with the structure of their derived categories.

4.1 A Noncommutative K3 Surface Associated with a Cubic Fourfold

A cubic fourfold is a hypersurface $X \subset \mathbb{P}^5$ of degree 3. By adjunction we have

$$K_X = -3H,$$

where H is the class of a hyperplane in $\mathbb{P}^5$. So, X is a Fano fourfold of index 3. Therefore, by Example 2.11 we have a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X(-2H), \mathcal{O}_X(-H), \mathcal{O}_X \rangle. \tag{16}$$

In fact, this is a maximal semiorthogonal decomposition.

Proposition 4.1 *The category $\mathcal{A}_X$ is a connected Calabi–Yau category of dimension* 2 *with Hochschild homology isomorphic to that of K*3 *surfaces. In particular, $\mathcal{A}_X$ is indecomposable and* (16) *is a maximal semiorthogonal decomposition.*

Proof The proof of the fact that $\mathsf{S}_{\mathcal{A}_X} \cong [2]$ is a bit technical, it can be found in [K04] and [K15b]. The Hochschild homology computation, on a contrary, is quite simple. By Lefschetz Hyperplane Theorem and Griffiths Residue Theorem, the Hodge diamond of X looks as

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Thus by HKR isomorphism we have $\mathsf{HH}_\bullet(\mathbf{D}(X)) = \mathsf{k}[2] \oplus \mathsf{k}^{25} \oplus \mathsf{k}[-2]$. Since $\mathcal{A}_X$ is the orthogonal complement of an exceptional triple in the category $\mathbf{D}(X)$, by additivity of Hochschild homology (Theorem 2.15) it follows that

$$\mathsf{HH}_\bullet(\mathcal{A}_X) = \mathsf{k}[2] \oplus \mathsf{k}^{22} \oplus \mathsf{k}[-2].$$

By HKR isomorphism this coincides with the dimensions of Hochschild homology of K3 surfaces.

Since $\mathcal{A}_X$ is a 2-Calabi–Yau category by Lemma 1.18 there is an isomorphism $\mathsf{HH}_i(\mathcal{A}_X) \cong \mathsf{HH}^{i+2}(\mathcal{A}_X)$. Therefore, from the above description of Hochschild homology it follows that $\mathsf{HH}^0(\mathcal{A}_X)$ is one-dimensional, i.e., the category $\mathcal{A}_X$ is connected (alternatively, $\mathsf{HH}^0(\mathcal{A}_X)$ can be easily computed via the technique of [K12]).

Indecomposability of $\mathcal{A}_X$ then follows from Proposition 2.21. The components generated by exceptional objects are indecomposable by Corollary 2.22. □

Being Calabi–Yau category of dimension 2, the nontrivial component $\mathcal{A}_X$ of $\mathbf{D}(X)$ can be considered as a noncommutative K3 surface. As we will see soon, for some special cubic fourfolds X there are equivalences $\mathcal{A}_X \cong \mathbf{D}(S)$ for appropriate K3 surfaces. Thus $\mathcal{A}_X$ is indeed a deformation of $\mathbf{D}(S)$ for a K3 surface S.

Clearly, if $\mathcal{A}_X \cong \mathbf{D}(S)$ then $\operatorname{gdim}(\mathcal{A}_X) = 2$ and pretending that the set of Griffiths components $\mathsf{Griff}(X)$ is well defined, it follows that it is empty. On the other hand, if $\mathcal{A}_X \not\cong \mathbf{D}(S)$ then $\operatorname{gdim}(\mathcal{A}_X) > 2$ by Conjecture 2.25, and so $\mathcal{A}_X$ is a Griffiths component. This motivates the following

Conjecture 4.2 ([K06b, Conjecture 3] and [K10, Conjecture 1.1]) A cubic fourfold X is rational if and only if there is a smooth projective K3 surface S and an equivalence $\mathcal{A}_X \cong \mathbf{D}(S)$.

We will show that this Conjecture agrees perfectly with the known cases of rational cubic fourfolds.

4.2 *Pfaffian Cubics*

Consider a 6×6 skew-symmetric matrix with entries being linear forms on $\mathbb{P}^5$. Its determinant is the square of a degree 3 homogeneous polynomial in coordinates on $\mathbb{P}^5$, called the Pfaffian of the matrix. The corresponding cubic hypersurface is called a pfaffian cubic.

In other words, if $\mathbb{P}^5 = \mathbb{P}(V)$ with $\dim V = 6$, a cubic hypersurface $X \subset \mathbb{P}(V)$ is pfaffian if there is another 6-dimensional vector space W and a linear map $\varphi : V \to \Lambda^2 W^\vee$ such that $X = \varphi^{-1}(\mathsf{Pf})$, where $\mathsf{Pf} \subset \mathbb{P}(\Lambda^2 W^\vee)$ is the locus of degenerate skew-forms. In what follows we will assume that X is not a cone (hence φ is an embedding), and that the subspace $\varphi(V) \subset \Lambda^2 W^\vee$ does not contain skew-forms of rank 2 (i.e., $\mathbb{P}(V) \cap \mathsf{Gr}(2, W^\vee) = \emptyset$). To unburden notation we consider V as a subspace of $\Lambda^2 W^\vee$.

Remark 4.3 There is a geometric way to characterize pfaffian cubics. First, assume a cubic X is pfaffian and is given by a subspace $V \subset \Lambda^2 W^\vee$. Choose a generic hyperplane $W_5 \subset W$ and consider the set $R_{W_5} \subset X$ of all degenerate skew-forms in V the kernel of which is contained in W_5. It is easy to see that the kernel of a rank 4 skew-form is contained in W_5 if and only if its restriction to W_5 is decomposable.

Note also that such restriction is never zero (since we assumed there are no rank 2 forms in V), hence the composition

$$\varphi_{W_5} : V \hookrightarrow \Lambda^2 W^\vee \twoheadrightarrow \Lambda^2 W_5^\vee$$

is an embedding. Therefore, $R_{W_5} = \varphi_{W_5}(\mathbb{P}(V)) \cap \mathsf{Gr}(2, W_5)$. For generic choice of W_5 this intersection is dimensionally transverse and smooth, hence R_{W_5} is a quintic del Pezzo surface in X. Vice versa, a cubic fourfold X containing a quintic del Pezzo surface is pfaffian (see [Bea00]).

If X is a pfaffian cubic fourfold, one can associate with X a K3 surface as follows. Consider the annihilator $V^\perp \subset \Lambda^2 W$ of $V \subset \Lambda^2 W^\vee$ and define

$$S = \mathsf{Gr}(2, W) \cap \mathbb{P}(V^\perp). \tag{17}$$

One can check the following

Lemma 4.4 *A pfaffian cubic X is smooth if and only if the corresponding intersection* (17) *is dimensionally transverse and smooth, i.e. S is a $K3$ surface.*

Any pfaffian cubic is rational. One of the ways to show this is the following.

By definition, the points of $X \subset \mathbb{P}(V)$ correspond to skew-forms on W of rank 4. Each such form has a two-dimensional kernel space $K \subset W$. Considered in a family, they form a rank 2 vector subbundle $\mathscr{K} \subset W \otimes \mathcal{O}_X$. Its projectivization $\mathbb{P}_X(\mathscr{K})$ comes with a canonical map to $\mathbb{P}(W)$. On the other hand, the K3 surface S defined by (17) is embedded into $\mathsf{Gr}(2, W)$, hence carries the restriction of the rank 2 tautological bundle of the Grassmanian, i.e. a subbundle $\mathcal{U} \subset W \otimes \mathcal{O}_S$. Its projectivization $\mathbb{P}_S(\mathcal{U})$ also comes with a canonical map to $\mathbb{P}(W)$.

Proposition 4.4 *The map $p : \mathbb{P}_X(\mathscr{K}) \to \mathbb{P}(W)$ is birational and the indeterminacy locus of the inverse map coincides with the image of the map $q : \mathbb{P}_S(\mathcal{U}) \to \mathbb{P}(W)$.*

Proof The proof is straightforward. Given $w \in \mathbb{P}(W)$ its preimage in $\mathbb{P}_X(\mathscr{K})$ is the set of all skew-forms in V containing w in the kernel, hence it is the intersection $\mathbb{P}(V) \cap \mathbb{P}(\Lambda^2 w^\perp) \subset \mathbb{P}(\Lambda^2 W^\vee)$. Typically it is a point, and if it is not a point then the map

$$V \hookrightarrow \Lambda^2 W^\vee \overset{w}{\twoheadrightarrow} w^\perp \subset W^\vee$$

has at least two-dimensional kernel, i.e. at most 4-dimensional image. If the image is contained in the subspace $U^\perp \subset w^\perp$ for a 2-dimensional subspace $U \subset W$, then U gives a point on S and (U, w) gives a point on $\mathbb{P}_S(\mathcal{U})$ which projects by q to w. □

Remark 4.6 One can show that if the K3 surface $S \subset \mathsf{Gr}(2, W) \subset \mathbb{P}(\Lambda^2 W)$ contains no lines, then the map $q : \mathbb{P}_S(\mathcal{U}) \to \mathbb{P}(W)$ is a closed embedding. If, however, S has a line L then $\mathcal{U}_{|L} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$ and $\mathbb{P}_L(\mathcal{U}_{|L}) \subset \mathbb{P}_S(\mathcal{U})$ is the Hirzebruch surface F_1. Denote by $\tilde{L} \subset \mathbb{P}_L(\mathcal{U}_{|L})$ its exceptional section. Then the map q contracts $\tilde{L}$ to a point, and does this for each line in S.

Consider the obtained diagram

$$\begin{tikzcd} & \mathbb{P}_X(\mathscr{K}) \arrow[dl] \arrow[dr, "p"] & & \mathbb{P}_S(\mathcal{U}) \arrow[dl, "q"'] \arrow[dr] & \\ X & & \mathbb{P}(W) & & S, \end{tikzcd}$$

choose a generic hyperplane $W_5 \subset W$, and consider the "induced diagram"

$$\begin{tikzcd} & X' := p^{-1}(\mathbb{P}(W_5)) \arrow[dl] \arrow[dr, "p"] & & q^{-1}(\mathbb{P}(W_5)) =: S' \arrow[dl, "q"'] \arrow[dr] & \\ X & & \mathbb{P}(W_5) & & S, \end{tikzcd}$$

Then it is easy to see that (for general choice of the hyperplane W_5) $X' \to X$ is the blowup with center in the quintic del Pezzo surface R_{W_5} (see Remark 4.3), and $S' \to S$ is the blowup with center in $R'_{W_5} := S \cap \mathsf{Gr}(2, W_5) = \mathbb{P}(V^{\perp}) \cap \mathsf{Gr}(2, W_5)$, which is a codimension 6 linear section of $\mathsf{Gr}(2, 5)$, i.e. just 5 points. Finally, the map $q : S' \to \mathbb{P}(W_5)$ is a closed embedding and the map p is the blowup with center in $q(S') \subset \mathbb{P}(W_5)$. Thus, the blowup of X in a quintic del Pezzo surface is isomorphic to the blowup of $\mathbb{P}(W_5) = \mathbb{P}^4$ in the surface S', isomorphic to the blowup of the K3 surface S in 5 points. In particular, X is rational.

To show the implication for the derived category one needs more work.

Theorem 4.7 *If X is a pfaffian cubic fourfold and S is the K3 surface defined by (17), then $\mathcal{A}_X \cong \mathbf{D}(S)$.*

A straightforward way to prove this is by considering the two semiorthogonal decompositions of X':

$$\begin{aligned} \mathbf{D}(X') &= \langle \mathbf{D}(R_{W_5}), \mathbf{D}(X) \rangle &&= \langle \mathbf{D}(R_{W_5}), \mathcal{A}_X, \mathcal{O}_{X'}(-2H), \mathcal{O}_{X'}(-H), \mathcal{O}_{X'} \rangle, \\ \mathbf{D}(X') &= \langle \mathbf{D}(S'), \mathbf{D}(\mathbb{P}(W_5)) \rangle &&= \langle \mathbf{D}(R'_{W_5}), \mathbf{D}(S), \mathbf{D}(\mathbb{P}(W_5)) \rangle. \end{aligned}$$

Taking into account that $\mathbf{D}(R_{W_5})$ is generated by an exceptional collection of length 7 (since R_{W_5} is isomorphic to the blowup of $\mathbb{P}^2$ in 4 points), we see that the first s.o.d. consists of ten exceptional objects and the category $\mathcal{A}_X$, while the second s.o.d. consists of ten exceptional objects and $\mathbf{D}(S)$. So, if we would have a Jordan–Hölder property, it would follow that $\mathcal{A}_X \cong \mathbf{D}(S)$. As the property is wrong, we should instead, find a sequence of mutations taking one s.o.d. to the other.

There is also a completely different proof of Theorem 4.7 based on homological projective duality (see [K07, K06b]).

Remark 4.8 In fact, the locus of pfaffian cubic fourfolds is a dense constructible subset in the divisor $\mathcal{C}_{14}$ in the moduli space of all cubic fourfolds, which consists of cubic fourfolds containing a quartic scroll or its degeneration, see [BRS].

4.3 Cubics with a Plane

Another interesting class of cubic fourfolds is formed by cubic fourfolds $X \subset \mathbb{P}(V)$ containing a plane. So, let $U_3 \subset V$ be a 3-dimensional subspace and $W_3 := V/U_3$. Assume that X contains $\mathbb{P}(U_3)$. Then the linear projection from $\mathbb{P}(U_3)$ defines a rational map $X \dashrightarrow \mathbb{P}(W_3)$. To resolve it we have to blowup the plane $\mathbb{P}(U_3)$ first. We will get the following diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ X & & \mathbb{P}(W_3) \end{tikzcd}$$

The map p here is a fibration with fibers being 2-dimensional quadrics, and the discriminant $D \subset \mathbb{P}(W_3)$ being a plane sextic curve. Assume for simplicity that X is sufficiently general, so that D is smooth. Then the double covering

$$S \to \mathbb{P}(W_3) \tag{18}$$

branched over D is a smooth K3 surface (and if D is mildly singular, which is always true for smooth X, then one should take S to be the minimal resolution of singularities of the double covering, see [Mos16]). Then S comes with a natural Severi–Brauer variety, defined as follows.

Lemma 4.9 *Let M be the Hilbert scheme of lines in the fibers of the morphism p. The natural morphism $M \to \mathbb{P}(W_3)$ factors as the composition $M \to S \to \mathbb{P}(W_3)$ of a $\mathbb{P}^1$-fibration $q : M \to S$, followed by the double covering $S \to \mathbb{P}(W_3)$.*

Proof By genericity assumption all fibers of p are either nondegenerate quadrics or quadrics of corank 1. For a nondegenerate fiber, the Hilbert scheme of lines is a disjoint union of two smooth conics. And for a degenerate fiber, it is just one smooth conic. Thus the Stein factorization of the map $M \to \mathbb{P}(W_3)$ is a composition of a nondegenerate conic bundle (i.e. a $\mathbb{P}^1$-fibration) with a double covering. Moreover, the double covering is ramified precisely in D, hence coincides with S. $\square$

The constructed Severi–Brauer variety $q : M \to S$ corresponds to an Azumaya algebra on S, which we denote $\mathcal{B}_0$. One can show that its pushforward to $\mathbb{P}(W_3)$ coincides with the sheaf of even parts of Clifford algebras of the quadric fibration $p : \widetilde{X} \to \mathbb{P}(W_3)$. The derived category $\mathbf{D}(S, \mathcal{B}_0)$ of sheaves of coherent $\mathcal{B}_0$-modules on S is known as a “twisted derived category” of S, or as a “twisted K3 surface”.

Proposition 4.10 *If X is a general cubic fourfold with a plane then $\mathcal{A}_X \cong \mathbf{D}(S, \mathcal{B}_0)$.*

Proof Again, one can consider two semiorthogonal decompositions of $\mathbf{D}(\widetilde{X})$:

$$\mathbf{D}(\widetilde{X}) = \langle \mathbf{D}(\mathbb{P}(U_3)), \mathcal{A}_X, \mathcal{O}_{\widetilde{X}}(-2H), \mathcal{O}_{\widetilde{X}}(-H), \mathcal{O}_{\widetilde{X}} \rangle,$$
$$\mathbf{D}(\widetilde{X}) = \langle \mathbf{D}(S, \mathcal{B}_0), \mathbf{D}(\mathbb{P}(W_3)) \otimes \mathcal{O}_{\widetilde{X}}(-H), \mathbf{D}(\mathbb{P}(W_3)) \rangle,$$

the first consists of six exceptional objects and the category $\mathcal{A}_X$, the second again consists of six exceptional objects and the category $\mathbf{D}(S, \mathcal{B}_0)$. An appropriate sequence of mutations (see [K10]) then identifies the categories $\mathcal{A}_X$ and $\mathbf{D}(S, \mathcal{B}_0)$. □

The following simple argument shows that the quadric fibration $\tilde{X} \to \mathbb{P}(W_3)$ is rational if and only if the twisting $\mathcal{B}_0$ of the K3 surface S is trivial.

Theorem 4.11 *The quadric fibration $\widetilde{X} \to \mathbb{P}(W_3)$ is rational over $\mathbb{P}(W_3)$ if and only if the Azumaya algebra $\mathcal{B}_0$ on S is Morita trivial.*

Proof Indeed, the first condition is equivalent to existence of a rational multisection of $p : \widetilde{X} \to \mathbb{P}(W_3)$ of odd degree, and the second is equivalent to existence of a rational multisection of $q : M \to S$ of odd degree. It remains to note that given a rational multisection of p of degree d and considering lines in the fibers of p intersecting it, we obtain a rational multisection of q also of degree d. Vice versa, given a rational multisection of q of degree d (i.e. d vertical and d horizontal lines in a general fiber of p) and considering the points of intersection of these vertical and horizontal lines, we obtain a rational multisection of p of degree d^2. □

4.4 Final Remarks

We conclude this section with a short (and incomplete) list of results and papers touching related subjects.

First, there is a notion of a "categorical representability" introduced by Bernardara and Bolognesi, see [BB13]. It is designed for the same purpose as the notion of a Griffiths component, but in a slightly different way.

Second, there is also a Hodge-theoretic approach to a characterization of rationality of cubic fourfolds, which takes its origins from Hassett's observations [Has00, Sect. 1]. In particular, there is a Hodge-theoretic version of Conjecture 4.2 (usually it is called Hassett's Conjecture, although Hassett himself was cautious about conjecturing this), see [AT14] for details. The relation between the two conjectures is discussed in [AT14] and [Add15]. A general discussion of the category $\mathcal{A}_X$ for a cubic fourfold can be found in [H15].

Finally, it is worth noting that there are two more (classes of) varieties which behave very similarly to cubic fourfolds. The first are the Fano fourfolds of index 2 and degree 10 sometimes called Gushel–Mukai fourfolds. Their birational properties are discussed in [DIM14] and [DK], and their derived categories in [KP]. The second class is formed by the so-called Küchle varieties of type (c5), see [Kuc95]. It was shown in *loc. cit.* that these varieties have a Hodge structure of a K3 surface in the middle cohomology, and in [K14] it was conjectured that their derived categories contain a noncommutative K3 category. However, so far nothing is known about their birational geometry.

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Classical Moduli Spaces and Rationality

Alessandro Verra

Abstract Moduli spaces and rational parametrizations of algebraic varieties have common roots. A rich album of moduli of special varieties was indeed collected by classical algebraic geometers and their (uni)rationality was studied. These were the origins for the study of a wider series of moduli spaces one could define as classical. These moduli spaces are parametrize several type of varieties which are often interacting: curves, abelian varieties, K3 surfaces. The course will focus on rational parametrizations of classical moduli spaces, building on concrete constructions and examples.

1 Introduction

These lectures aim to study, even in a historical perspective, the interplay and the several links between two notions which are at the core of Algebraic Geometry, from its origins to modern times. The key words to recognize these notions are *Rationality* and *Moduli*.

Rationality is a classical notion. As we all know, an algebraic variety X over a field k is rational if there exists a birational map

$$f : k^d \to X$$

with $d = dim\ X$. The notion reflects the original attempt to study some algebraic varieties, the rational ones, via invertible parametric equations defined by rational functions.

Given an algebraic variety X the *rationality problem for* X, when it is meaningful, is the problem of understanding wether X is rational or not. In most of the relevant cases this is an outstanding problem coming from classical heritage. Therefore the rationality problems, for some important classes of algebraic varieties, often are

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guiding themes, marking the history of Algebraic Geometry. As a consequence of this fact many related notions came to light and are of comparable importance.

Let us recall the basic ones and some relevant properties. We will often consider the notion of *unirational variety*. This originates from the most natural instance to weaken the notion of rationality. Indeed we may want simply to construct a dominant separable rational map

$$f : k^n \to X$$

not necessarily birational. We say that f is a *rational parametrization of* X.

Definition 1.1 An algebraic variety X is unirational if there exists a rational parametrization $f : k^n \to X$.

In spite of their simple definitions, these classical notions are intrinsically hard. Studying the rationality, or unirationality, of algebraic varieties has proven to be a very nasty problem, even for the tools of contemporary Algebraic Geometry. This was often remarked in the modern times by many authors, see Kollar's article [K] about. In order to have a more doable notion, that of *rational connectedness* became central in the last three decades.

Definition 1.2 X is rationally connected if there exists a non empty open subset U of X such that: $\forall\, x, y \in U\ \exists f_{x,y} : k \to X \,/\, x, y \in f(k)$.

An important example where rational connectedness appears as more doable is the following. Assume k is the complex field and consider an irreducible family $f : \mathcal{X} \to B$ of smooth irreducible complex projective varieties. Then the next statement is well known: the property of a fibre of being rationally connected extends from one to all the fibres of f.

The same statement is instead unproven if we replace the word rationally connected by the words rational or unirational. Actually such a statement is expected to be false. If so this would be a crucial difference between the notions considered. For completeness let us also recall that[1]

Definition 1.3 X is uniruled if there exists a non empty open subset U of X such that: $\forall\, x \in U,\ \exists$ a non constant $f_x : k \to X \,/\, x \in f(k)$.

In view of the contents of these lectures we will work from now on over the complex field. It is easy to see that

$$\textit{Rational} \overset{(1)}{\Rightarrow} \textit{unirational} \overset{(2)}{\Rightarrow} \textit{rationally connected} \overset{(3)}{\Rightarrow} \textit{uniruled.} \overset{(4)}{\Rightarrow} \textit{kod}(X) = -\infty$$

[1] A first counterexample to this outstanding problem has been finally produced by B. Hassett, A. Pirutka and Y. Tschinkel in 2016. See: *Stable rationality of quadric surfaces bundles over surfaces*, arXiv1603.09262

where $kod(X)$ is the *Kodaira dimension* of X.[2] Remarking that the inverse of implication (3) is false is a very easy exercise: just consider as a counterexample $X = Y \times \mathbf{P}^1$, where Y is not rationally connected.

On the other hand the remaining problems, concerning the inversion of the other implications, have a more than centennial history and a prominent place in Algebraic Geometry, cfr. [EM, Chaps. 6–10]. Inverting (1) is the classical Lüroth problem, which is false in dimension ≥ 3. As is well known this is due to the famous results of Artin and Mumford, Clemens and Griffiths, Manin and Iskovskih in the Seventies of last century. It is reconsidered elsewhere in this volume. Wether (2) is invertible or not is a completely open question:

Does rational connectedness imply unirationality?

This could be considered, in some sense, the contemporary version of the classical Lüroth problem. That (4) is invertible is a well known conjecture due to Mumford. It is time to consider the other topic of these lectures.

Moduli, in the sense of moduli space, is a classical notion as well. Riemann used the word Moduli, for algebraic curves, with the same meaning of today. Under some circumstances the moduli space $\mathcal{M}$ of a family of algebraic varieties X is dominated by the space of coefficients of a *general* system of polynomial equations of fixed type. Hence $\mathcal{M}$ is unirational in this case.

For instance every hyperelliptic curve H of genus $g \geq 2$ is uniquely defined as a finite double cover $\pi : H \to \mathbf{P}^1$ and π is uniquely reconstructed from its branch locus $b \in \mathbf{P}^{2g+2} := |\mathcal{O}_{\mathbf{P}^1}(2g+2)|$. Therefore the assignment $b \longrightarrow H$ induces a dominant rational map $m : \mathbf{P}^{2g+2} \to \mathcal{H}_g$, where $\mathcal{H}_g$ is the moduli space of hyperelliptic curves of genus g. This implies that $\mathcal{H}_g$ is unirational for every genus g. Actually, a modern result in Invariant Theory due to Katsylo says that $\mathcal{H}_g$ is even rational [K5].

The latter example highlights the so many possible relations between questions related to rationality and the natural search of suitable parametrizations for some moduli spaces. As remarked, these relations closely follow the evolution of Algebraic Geometry from nineteenth century to present times. This is specially true for the moduli space $\mathcal{M}_g$, of curves of genus g, and for other moduli spaces, quite related to curves, more recently appeared in this long history. In some sense we could consider these spaces as the classical ones. Here is a list, according to author's preferences:

- $\mathcal{M}_g$, *moduli of curves of genus* g;
- $Pic_{d,g}$, *universal Picard variety over* $\mathcal{M}_g$;
- $\mathcal{R}_g$, *moduli of Prym curves of genus* g;

[2] We recall that $kod(X) := \min\{\dim f_m(X'),\ m \geq 1\}$. Here X' is a complete, smooth birational model of X. Moreover f_m is the map defined by the linear system of pluricanonical divisors $P_m := \mathbf{P}H^0(det(\Omega^1_{X'})^{\otimes m})$. If P_m is empty for each $m \geq 1$ one puts $kod(X) := -\infty$.

- $\mathcal{S}_g^+/\mathcal{S}_g^-$, *moduli of even/odd spin curves of genus* g;
- $\mathcal{A}_g$, *moduli of p.p. abelian varieties of dimension* g;
- $\mathcal{F}_g$, *moduli of polarized K3 surfaces of genus* g.

In these lectures we review a variety of results on some of these spaces and their interplay with rationality, for this reason we specially concentrate on the cases of low genus g. We focus in particular on the next themes:

- *How much the rationality of these spaces is extended*,
- *Uniruledness/ Unirationality/ Rationality*,
- *Transition to non negative Kodaira dimension as g grows up.*

Classical geometric constructions grow in abundance around the previous spaces, revealing their shape and their intricate connections. The contents of the forthcoming lectures are organized as follows.

In Sect. 2 of these notes we study the families $\mathcal{V}_{d,g}$ of nodal plane curves of degree d and genus g. Starting from Severi's conjecture on the (uni)rationality of $\mathcal{M}_g$ for any g, and coming to the results of Harris and Mumford which disprove it, we describe results and attempts in order to parametrize $\mathcal{M}_g$ by a rational family of linear systems of nodal plane curves.

Then we study, by examples and in low genus, the effective realizations of non isotrivial pencils of curves with general moduli on various types of surfaces. In particular we use these examples to revisit the slope conjecture, for the cone of effective divisors of the moduli of stable curves $\overline{\mathcal{M}}_g$, and its counterexamples.

In Sect. 3 we study the moduli spaces $\mathcal{R}_g$ of Prym curves and the Prym map $P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Then we prove the unirationality of $\mathcal{R}_g$ for $g \leq 6$. We outline a new method of proof used in the very recent papers [FV4, FV5]. We use families of nodal conic bundles over the plane, instead of families of nodal plane curves, to construct a rational parametrization of $\mathcal{R}_g$. We also discuss the known results on the rationality of $\mathcal{R}_g$ for $g \leq 4$.

In Sect. 4 we profit of these results to discuss the unirationality of $\mathcal{A}_p$. The unirationality of $\mathcal{A}_p$ is granted by the unirationality of $\mathcal{R}_{p+1}$ and the dominance of the Prym map P_{p+1} for $p \leq 5$. An interesting object here is the universal Prym, that is the pull back by P_{p+1} of the universal principally polarized abelian variety over $\mathcal{A}_p$. We construct an effective rational parametrization of the universal Prym $\mathcal{P}_5$.

Finally we discuss the slope of a toroidal compactification of $\mathcal{A}_p$, $p \leq 6$. Let $\overline{\mathcal{A}}_p$ be the perfect cone compactification of $\mathcal{A}_p$, it turns out that, for $p = 6$, the boundary divisor D of $\overline{\mathcal{A}}_6$ is dominated by the universal Prym $\mathcal{P}_5$. We use its effective parametrization, and a sweeping family of rational curves on it, to compute a lower bound for the slope of $\overline{\mathcal{A}}_6$, outlining the main technical details from [FV4].

In Sect. 5 we deal with the theme of curves and K3 surfaces. Mukai constructions of general K3 surfaces of genus $g \leq 11$, and hence of general canonical curves for $g \neq 10$, are considered. We apply these to prove the unirationality of the universal Picard variety $Pic_{d,g}$ for $g \leq 9$, we will use it to obtain further unirationality results for $\mathcal{M}_g$. The transition of the Kodaira dimension of $Pic_{d,g}$, from $-\infty$ to the maximal one, is also discussed.

Then we concentrate on the family of special K3 surfaces S, having Picard number at least two, embedded in $\mathbf{P}^{g-2}$ by a genus $g-2$ polarization and containing a smooth paracanonical curve C of genus g. This is the starting point for going back to Prym curves (C, η) of genus g and discuss when $\mathcal{O}_C(1)$ is the Prym canonical line bundle $\omega_C \otimes \eta$.

In the family of K3 surfaces S we have the family of Nikulin surfaces, where the entire linear system $|C|$ consists of curves D such that $\mathcal{O}_D(1)$ is Prym canonical. We discuss this special situation and some interesting analogies between the families of curves C of genus $g \leq 7$ contained in a general Nikulin surface S and the families of curves of genus $g \leq 11$ contained in a general K3 surface.

An outcome of the discussion is the proof that the Prym moduli space $\mathcal{R}_7$ is unirational, cfr. [FV6]. Finally, in Sect. 6, we go back to $\mathcal{M}_g$. Relying on the previous results, we outline from [Ve1] the proof that $\mathcal{M}_g$ is unirational for $g \leq 14$.

2 $\mathcal{M}_g$ and the Rationality

2.1 *Nodal Plane Sextics*

We begin our tour of classical moduli spaces and related geometric constructions from moduli of curves. If we want to go back to the classical point of view, adding some historical perspectives, it is natural to consider at first plane curves. Any algebraic variety V of dimension d is birational to a hypersurface in $\mathbf{P}^{d+1}$ and the study of these birational models of V was considered a natural option to understand V: curves in the plane, surfaces in the space and so on.

In 1915 Severi publishes the paper "Sulla classificazione delle curve algebriche e sul teorema di esistenza di Riemann," an exposition of his recent results on the birational classification of algebraic curves. The next sentence from it is the starting point of a long history:

> Ritengo probabile che la varietá H sia razionale o quanto meno che sia riferibile ad un' involuzione di gruppi di punti in uno spazio lineare S_{3p-3}; o, in altri termini, che nell' equazione di una curva piana di genere p (e per esempio dell' ordine $p+1$) i moduli si possano far comparire razionalmente.

See [S], in the text H is the moduli space of curves $\mathcal{M}_g$. The conjecture of Severi is that $\mathcal{M}_g$ is "probably rational" or at least unirational. The idea is that there exists an irreducible family of plane curves, to be written in affine coordinates as

$$\sum_{0 \leq i,j \leq d} f_{ij} X^i Y^j = 0,$$

so that:

- *its general element is birational to a curve of genus g*,
- *the f_{ij}'s are rational functions of* $3g-3$ *parameters*,
- *the corresponding natural map* $f : \mathbf{C}^{3g-3} \to \mathcal{M}_g$ *is dominant.*[3]

In other words the claim is that there exists a unirational variety

$$\mathbb{P} \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$$

of possibly singular plane curves of degree d such that f is dominant. In the same paper a proof is given that such a family exists for $g \leq 10$.

Now we postpone for a while the discussion of this result to concentrate on the simplest possible case, namely when $\mathbb{P}$ is a linear space. In this case $\mathbb{P}$ is a linear system of plane curves, possibly singular at some of the base points. The first question we want to consider, even in a more general framework than $\mathbf{P}^2$, is the following.

Question 2.1 *Let* $\mathbb{P}$ *be a linear system of curves of geometric genus g on a smooth surface P, when the natural map* $m : \mathbb{P} \to \mathcal{M}_g$ *is dominant?*

The complete answer follows from a theorem of Castelnuovo, with the contribution of Beniamino Segre and later of Arbarello [Ar, Se1]. By resolution of indeterminacy there exists a birational morphism

$$\sigma : S \to P$$

such that S is smooth and the strict transform of $\mathbb{P}$ is a base point free linear system $|C|$ of smooth, integral curves of genus g. Then the answer to Question 2.1 is provided by the next theorem.

Theorem 2.2 *Let* $|C|$ *be as above. Assume that the natural map*

$$m : |C| \to \mathcal{M}_g$$

is dominant, then S is rational and $g \leq 6$.

Proof It is not restrictive to assume that $|C|$ is base point free. Since $|C| \to \mathcal{M}_g$ is dominant we have $\dim\ |C| \geq 3g-3$. This implies that $\mathcal{O}_C(C)$ is not special. Then, by Riemann Roch, it follows $C^2 \geq 4g-4$ and we have $CK_S \leq -2g+2$ by adjunction formula. Since $|C|$ has no fixed components, it follows that $|mK_S| = \emptyset$ for $m \geq 1$. Hence S is ruled and birational to $R \times \mathbf{P}^1$. In particular C admits a finite map $C \to R$. Since the curves of $|C|$ have general moduli, this is impossible unless R is rational. Hence S is rational. Now let $g \geq 10$, we observe that $\dim |C| \geq 3g-3 \geq 2g+7$. But then a well known theorem of Castelnuovo, on linear systems on a rational surface,

[3]Unless differently stated, we assume $g \geq 2$ to simplify the exposition.

implies that the elements of $|C|$ are hyperelliptic [C1]. This is a contradiction, hence the theorem follows for $g \geq 10$. The cases $g = 7, 8, 9$ were excluded by Beniamino Segre in [Se1]. $\square$

Let $g \leq 6$, one can easily show that $\mathbb{P}$ can be chosen so that $d = 6$ and its base points are ordinary nodes for a general $\Gamma \in \mathbb{P}$. For $g \geq 2$ these are also in general position in $\mathbf{P}^2$. In some sense the case of plane sextics of genus $g \leq 6$ will become a guiding example for these lectures.

2.2 Rationality of $\mathcal{M}_6$

In spite of the centennial history of moduli of curves, the rationality of $\mathcal{M}_g$ remains unsettled, as an outstanding problem, for most of the values of g where $\mathcal{M}_g$ is known to be unirational or uniruled. Presently this means $g \leq 16$. The rationality of $\mathcal{M}_g$ is actually established for $g \leq 6$.

The known rationality results are mainly related to Invariant Theory, starting from the classical description

$$\mathcal{M}_1 = \mathfrak{h}/SL(2, \mathbb{Z}).$$

Here the action of $SL(2, \mathbb{Z})$ on the Siegel upper half plane $\mathfrak{h}$ is

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \forall \tau \in \mathfrak{h}\ ,\ \forall \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in SL(2, \mathbb{Z})$$

and $\mathcal{M}_1$ is actually the affine line. As it is well known this rationality result goes back to Weierstrass and to the Weierstrass form of the equation of a plane cubic. The rationality of $\mathcal{M}_2$ was proven in 1960 by Igusa [I]. Later the proof of the rationality of $\mathcal{M}_4$ and $\mathcal{M}_6$ came around 1985. This is due to Shepherd-Barron [SB] and [SB1]. Finally the rationality of $\mathcal{M}_5$ and of $\mathcal{M}_3$ is due to Katsylo [K1] and [K3]. See also [Bo, K4].

The rationality of $\mathcal{M}_6$ is related to plane sextics with four-nodes. In turn, as we are going to see in these lectures, the family of nodal plane sextics of genus $g \leq 6$ is also related to the unirationality of the Prym moduli spaces $\mathcal{R}_g$ for $g \leq 6$ [FV4].

Let $\mathbb{P}$ be the linear system of plane sextics having multiplicity ≥ 2 at the fundamental points $F_1 = (1:0:0)$, $F_2 = (0:1:0)$, $F_3 = (0:0:1)$, $F_4 = (1:1:1)$ of $\mathbf{P}^2$. Then a general $\Gamma \in \mathbb{P}$ is birational to a genus six curve. We know from the previous section that the moduli map

$$m : \mathbb{P} \to \mathcal{M}_6$$

is dominant. Hence m is generically finite, since $\dim \mathbb{P} = \dim \mathcal{M}_6$. What is the degree of m? To answer this question consider the normalization

$$\nu : C \to \Gamma$$

of a general $\Gamma \in \mathbb{P}$. At first we notice that $\nu^*\mathcal{O}_\Gamma(1)$ is an element of

$$W^2_6(C) := \{L \in Pic^6(C) \;/\; h^0(L) \geq 3\} \subset Pic^6 C.$$

It is well known that the cardinality of $W^2_6(C)$ is five and we can easily recover all the elements of this set from Γ. One is $\nu^*\mathcal{O}_\Gamma(1)$. Moreover let

$$Z \subset \{F_1 \dots F_4\}$$

be a set of three points, then $L_Z = \nu^*\mathcal{O}_\Gamma(2) \otimes \mathcal{O}_C(-\nu^*Z)$ is also an element of $W^2_6(C)$. In particular the linear system $|L_Z|$ is just obtained by taking the strict transform by ν of the net of conics through Z.

Next we remark that two elements $\Gamma_1, \Gamma_2 \in \mathbb{P}$ are birational to the same C and defined by the same $L \in W^2_6(C)$ iff there exists $\alpha \in \mathfrak{s}_4 \subset Aut\, \mathbb{P}^2$ such that $\alpha(\Gamma_1) = \Gamma_2$.

Here we denote by $\mathfrak{s}_4$ the stabilizer of $\{F_1 \dots F_4\}$ in $Aut\, \mathbb{P}^2$, which is a copy of the symmetric group in four letters. Since the cardinality of $W^2_6(C)$ is five we conclude that:

Proposition 2.3 *m has degree* 120.

Is this degree related to an action of the symmetric group $\mathfrak{s}_5$ on $\mathbb{P}$ so that m is the quotient map of this action?

The answer is positive: in the Cremona group of $\mathbf{P}^2$ we can consider the subgroup generated by $\mathfrak{s}_4$ and by the quadratic transformations

$$q_Z : \mathbf{P}^2 \to \mathbf{P}^2$$

centered at the subsets of three points $Z \subset \{F_1 \dots F_4\}$. This is actually a copy of $\mathfrak{s}_5$ and we will denote it in the same way. Notice that the strict transform of $\mathbb{P}$ by q_Z is $\mathbb{P}$, hence $\mathfrak{s}_5$ is exactly the subgroup of the Cremona group of $\mathbf{P}^2$ leaving $\mathbb{P}$ invariant.

Equivalently we can rephrase the previous construction in terms of the surface obtained by blowing up $\mathbf{P}^2$ along $\{F_1 \dots F_4\}$. Let

$$\sigma : S \to \mathbf{P}^2$$

be such a blowing up. Then S is a quintic Del Pezzo surface and, moreover, the strict transform of $\mathbb{P}$ by σ is exactly the linear system

$$|-2K_S|,$$

where K_S is a canonical divisor. As is well known, the anticanonical linear system $|-K_S|$ defines an embedding $S \subset \mathbf{P}^5$ as a quintic Del Pezzo surface and S is the unique smooth quintic Del Pezzo up to projective equivalence.

The action of $\mathfrak{s}_5$ on $\mathbf{P}^2$ lifts to an action of $\mathfrak{s}_5$ on S as a group of biregular automorphisms and it is known, and easy to see, that

$$\mathfrak{s}_5 = Aut\, S.$$

The conclusion is now immediate:

Theorem 2.4 $\mathcal{M}_6 \cong \mathbb{P}/\mathfrak{s}_5 \cong |-2K_S|/Aut\, S.$

This situation is considered in [SB]. Building on representation theory of $\mathfrak{s}_5$, the author then shows the following result.

Theorem 2.5 *$|-2K_S|/Aut\, S$, and hence $\mathcal{M}_6$, is rational.*

Let us sketch the proof very briefly. The symmetric group $\mathfrak{s}_5 = Aut\, S$ acts on the vector space $V := H^0(\mathcal{O}_S(-K_S))$ so that V is a representation of degree 6 of $\mathfrak{s}_5$. This is the unique irreducible representation of degree 6 of $\mathfrak{s}_5$. Passing to $Sym^2 V$ one has

$$Sym^2 V = I_2 \oplus H^0(\mathcal{O}_S(-2K_S))$$

where I_2 is Kernel of the multiplication map $Sym^2 V \to H^0(\mathcal{O}_S(-2K_S))$. Clearly the summands are representations of $\mathfrak{s}_5$. For $H^0(\mathcal{O}_S(-2K_S))$ the point is to show that it is isomorphic to

$$\mathbf{1} \oplus \phi \oplus \chi \oplus \chi' \oplus \sigma.$$

Here $\mathbf{1}$ and σ denote the trivial and sign representations of degree 1 and ϕ is the standard representation of degree 4. Moreover χ, χ' are irreducible of degree 5 and either $\chi = \chi'$ or $\chi' = \chi \otimes \sigma$.

Blowing up the Kernel of the projection $\mathbf{1} \oplus \phi \oplus \chi \oplus \chi' \oplus \sigma \to \phi$ one obtains a vector bundle $\mathcal{E} \to \phi$ of rank 12. Notice that $\mathfrak{s}_5$ acts freely on an invariant open set $U \subset \phi$ and that the action of $\mathfrak{s}_5$ is linearized on $\mathcal{E}$. Therefore $\mathcal{E}$ descends to a vector bundle $\overline{\mathcal{E}} \to U/\mathfrak{s}_5$ and, moreover, it follows that

$$\mathbf{P}(\overline{\mathcal{E}}) \ \cong\ \mathbf{P}^{11} \times U/\mathfrak{s}_5 \ \cong |-2K_S|/\mathfrak{s}_5 \ \cong \mathcal{M}_6.$$

It remains to show that $U/\mathfrak{s}_5$ is rational, but this just follows from the theorem of symmetric functions.

2.3 Nodal Plane Curves

As a second step, of our investigation on linear systems of plane curves dominating $\mathcal{M}_g$, we replace a single linear system by a family. We say that

$$\mathbb{P}_T \subset T \times |\mathcal{O}_{\mathbb{P}^2}(d)|$$

is a family of linear systems of curves of degree d and genus g if:

(1) *T is a integral variety in the Grassmannian of r-spaces of $|\mathcal{O}_{\mathbb{P}^2}(d)|$,*
(2) *the first projection $p_1 : \mathbb{P}_T \to T$ is the universal r-space over T,*
(3) *a general $\Gamma \in p_2(\mathbb{P}_T)$ is an integral curve of genus g.*

Here p_2 denotes the second projection $\mathbb{P}_T \to |\mathcal{O}_{\mathbf{P}^2}(d)|$, we will set

$$\mathbb{P} := p_2(\mathbb{P}_T) \subset |\mathcal{O}_{\mathbb{P}^2}(d)|.$$

Clearly we have

$$\mathbb{P} = \bigcup_{t \in T} \mathbb{P}_t,$$

where we denote by $\mathbb{P}_t$ the fibre of p_1 at t. $\mathbb{P}_t$ is an r-dimensional linear system of plane curves of degree d and geometric genus g.

Definition 2.6 If $r \geq 1$ we say that $\mathbb{P}$ is a scroll of plane curves of degree d and genus g.

Assume that $\mathbb{P}$ is such a scroll and that the natural moduli map

$$m : \mathbb{P} \to \mathcal{M}_g$$

is dominant. Then it turns out that $m/\mathbb{P}_t$ is not constant for a general t, see Lemma 2.9, and hence a general point of $\mathcal{M}_g$ belongs to a unirational variety $m(\mathbb{P}_t)$. This implies that $\mathcal{M}_g$ is uniruled. Not so differently: let T be unirational and m dominant. Then $\mathbb{P}_T$, hence $\mathcal{M}_g$, are unirational.

The classical attempts to prove the unirationality or the uniruledness of $\mathcal{M}_g$ are in this spirit. To summarize it, we raise the following:

Question 2.7 *Does there exist a scroll $\mathbb{P}$ of plane curves of degree d and genus g such that the moduli map $m : \mathbb{P} \to \mathcal{M}_g$ is dominant?*

We know nowadays that no such a $\mathbb{P}$ exists at least from $g \geq 22$. Indeed $\mathcal{M}_g$ is not uniruled for $g \geq 22$ because its Kodaira dimension is not $-\infty$. We also point out that the existence of $\mathbb{P}$ is a sufficient condition for the uniruledness. But it is not at all necessary, as we will see in Example 2.32. Let's start a sketch of the classical approaches of Severi and Beniamino Segre.

Let $t \in T$, we consider the base scheme B_t of $\mathbb{P}_t$ and denote its reduced scheme as $Z_t := B_{t,red}$. Up to replacing T by a non empty open subset, we will assume that T is smooth and that the family

$$Z := \{(x, t) \in \mathbf{P}^2 \times T \ / \ x \in Z_t\}$$

is a flat family of smooth 0-dimensional schemes of length b.

Lemma 2.8 *There exists a commutative diagram*

$$\begin{array}{ccc} T \times \mathbf{P}^2 & \xrightarrow{F} & T \times \mathbf{P}^2 \\ p \downarrow & & p \downarrow \\ T & \xrightarrow{id_T} & T \end{array}$$

such that F is birational and, for a general $t \in T$, the strict transform of $\mathbb{P}_t$ by $F/\{t\} \times \mathbf{P}^2$ is a linear system of curves with ordinary multiple points.

The proof follows in a standard way from Noether's theorem on reduction of the singularities of a plane curve to ordinary ones by a Cremona transformation, we omit it. We assume from now on that a general $\Gamma \in \mathbb{P}$ has ordinary singularities. For general $t \in T$ and $\Gamma \in \mathbb{P}_t$ we then have

$$g = \binom{d-1}{2} - \sum_{i=1\dots b} \binom{m_i}{2},$$

where m_i is the multiplicity of Γ at the base point $x_i \in Z_t$. Let

$$\sigma : S \to \mathbf{P}^2$$

be the blowing up of Z_t. Then the strict transform of Γ is a smooth, integral curve C of genus g on the smooth rational surface S. From the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

it follows $h^1(\mathcal{O}_S(C)) = h^1(\mathcal{O}_C(C))$, moreover we have

$$\dim |C| = h^0(\mathcal{O}_C(C)).$$

More in general let $C \subset S$, where S is any smooth surface not birational to $C \times \mathbf{P}^1$. Consider the moduli map $m : |C| \to \mathcal{M}_g$, then we have:

Lemma 2.9 *If* $\dim |C| \geq 1$ *and C is general then m is not constant.*

Proof Assume $g \geq 3$ and that C moves in a pencil $P \subset |C|$ whose general member has constant moduli. Since C is general we can assume $Aut\ C = 1$. Hence we can define the birational map $\phi : S \to C \times P$ such that $\phi(x) = (x, z)$, where $x \in C_z = C$ and C_z is the unique element of P passing through x. This implies that S is birational to $C \times \mathbf{P}^1$: a contradiction. The easy extension of this argument to the case $g = 2$ is left to the reader. □

We can finally start our search for a scroll $\mathbb{P}$, outlining the classical approach. Let $\mathbb{P}'_t \subset |C|$ be the strict transform of $\mathbb{P}_t$ by $\sigma : S \to \mathbf{P}^2$. Since a general $\Gamma \in \mathbb{P}_t$ has ordinary multiple points it follows that

$$C^2 = d^2 - m_1^2 - \cdots - m_b^2.$$

Since C is integral and $\dim |C| \geq 1$, it is elementary but crucial that

$$C^2 \geq 0.$$

From now on we denote by δ the number of singular points of a general $\Gamma \in \mathbb{P}$. Clearly we have $\delta \leq b$, hence it follows

$$d^2 - m_1^2 - \cdots - m_\delta^2 \geq 0,$$

where m_i is the multiplicity of the i-th singular point of Γ. The starting point of Severi is the case where Γ is a *nodal plane curve*, that is, the case

$$m_1 = \cdots = m_\delta = 2.$$

Analyzing this case, Severi obtains the following result:

Theorem 2.10 *$\mathcal{M}_g$ is unirational for $g \leq 10$.*

Severi's arguments rely on the irreducibility of $\mathcal{M}_g$ and on Brill-Noether theory, two established results for the usually accepted standards of that time. Further precision was however needed, as we will see. Let us discuss this matter, after a few reminders on Brill-Noether theory [ACGH, Chap. IV].

Definition 2.11 The Brill-Noether loci of a curve C are the loci

$$W_d^r(C) := \{L \in Pic^d(C) \ / \ h^0(L) \geq r + 1\}.$$

$W_d^r(C)$ has a natural structure of determinantal scheme. The tangent space at the element $L \in W_d^r(C) - W_d^{r+1}(C)$ is determined by the Petri map

$$\mu : H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C).$$

Indeed it is the orthogonal space $Im\ \mu^\perp \subset H^0(\omega_C)^*$. If μ is injective one computes that $dim\ Im\ \mu^\perp$ is the Brill-Noether number

$$\rho(g, r, d) := g - (r + 1)(r + g - d),$$

where g is the genus of C. The next theorems are nowadays well known.

Theorem 2.12 *Let C be a general curve of genus g, then:*

(1) *$W_d^r(C)$ is not empty iff $\rho(g, r, d) \geq 0$,*
(2) *let $\rho(g, r, d) \geq 0$ then $dim\ W_d^r(C) = \rho(g, r, d)$.*

Theorem 2.13 *Let C be a general curve and L general in $W^r_d(C)$, then:*

(1) *if $r \geq 3$ then the line bundle L is very ample,*
(2) *if $r = 2$ then $|L|$ defines a generically injective map in $\mathbf{P}^2$ whose image is a nodal curve.*

Let $\Gamma \subset \mathbf{P}^2$ be a plane curve of degree d and geometric genus g. Let $\nu : C \to \Gamma$ be its normalization map. Applying Theorem 2.12 (1) it follows:

Proposition 2.14 *If the curve C has general moduli then*

$$g \leq \frac{3}{2}d - 3.$$

To describe the geometry of $\mathcal{M}_g$, Severi considers the families

$$\mathcal{V}_{d,g} \subset |\mathcal{O}_{\mathbf{P}^2}(d)|$$

of all integral and nodal plane curves of degree d and genus g. Let

$$H_\delta$$

be the open set, in the Hilbert scheme of δ points of $\mathbf{P}^2$, whose elements are smooth. Then $\mathcal{V}_{d,g}$ is endowed with the morphism

$$h : \mathcal{V}_{d,g} \to H_\delta$$

sending Γ to $h(\Gamma) := Sing\,\Gamma$, where $\delta = \binom{d-1}{2} - g$. The families of curves $\mathcal{V}_{d,g}$ are well known as the *Severi varieties* of nodal plane curves.

Now assume that a scroll $\mathbb{P}$, of plane curves of degree d and genus g, dominates $\mathcal{M}_g$ via the moduli map. Then we have

$$\mathbb{P} \subseteq \overline{\mathcal{V}}_{d,g}$$

where $\overline{\mathcal{V}}_{d,g}$ denotes the closure of $\mathcal{V}_{d,g}$. Notice also that the fibres of the map $f : h/\mathbb{P} : \mathbb{P} \to H_\delta$ are the linear systems $\mathbb{P}_t$ already considered. Therefore we can replace T by $h(\mathbb{P})$ and $\mathbb{P}$ by $\overline{\mathcal{V}}_{d,g}$ and directly study the latter one.

However the study of $\mathcal{V}_{d,g}$ involves some delicate questions, which were left unsettled for long time after Severi. These are related to his unirationality result for $\mathcal{M}_g$, $g \leq 10$. A main question, now solved, concerns the irreducibility of $\mathcal{V}_{d,g}$, claimed in [S1] and finally proven by Harris in [H2]:

Theorem 2.15 *Let $\mathcal{V}_{d,g}$ be the Severi variety of integral nodal curves of degree d and genus $g = \binom{d-1}{2} - \delta$. Then $\mathcal{V}_{d,g}$ is integral, smooth and of codimension δ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$.*

Once this theorem is granted, a first very natural issue is to consider the case where both the maps

$$h : \overline{\mathcal{V}}_{d,g} \to H_\delta \text{ and } m : \overline{\mathcal{V}}_{d,g} \to \mathcal{M}_g$$

are dominant. A first reason for doing this is that H_δ is rational. If h is dominant then h defines a $\mathbf{P}^r$-bundle structure over an open subset of H_δ. Then, since it is irreducible, $\mathcal{V}_{d,g}$ is birational to $H_\delta \times \mathbf{P}^r$. Hence $\mathcal{M}_g$ is unirational if m is dominant.

We know from Brill-Noether theory that in this case $g \leq \frac{3}{2}d - 3$. So we have to compare this inequality and the condition that h is dominant. For a general $\Gamma \in \mathcal{V}_{d,g}$, let $Z := Sing\ \Gamma$ and $\mathcal{I}_Z$ its ideal sheaf. We can use deformation theory for the family of nodal plane curves $\mathcal{V}_{d,g}$ as it is given in [Ser2, 4.7]. In particular we have that $h^0(\mathcal{I}_Z(d) - h^0(\mathcal{I}^2_Z(d))$ is the rank of the tangent map dh at Γ. Now assume that h is dominant. Then Z is general in H_δ and hence $h^0(\mathcal{I}_Z(d)) = h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - \delta$. Furthermore dh_Γ is surjective and $h^0(\mathcal{I}^2_Z(d)) \geq 1$ because $\Gamma \in |\mathcal{I}^2_Z(d)|$. Hence we have

$$h^0(\mathcal{I}^2_Z(d)) = h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - 3\delta = \binom{d+2}{2} - 3\delta \geq 1.$$

Assume h and m are both dominant. Since $\delta = \binom{d-1}{2} - g$ we deduce

$$g \leq \frac{3}{2}d - 3 \text{ and } \frac{d^2}{3} - 2d + 1 \leq g.$$

Then, relying on the previous theorems, one easily concludes that

Theorem 2.16 *m and h are dominant iff* $g \leq 10$, $d \leq 9$ *and* $g \leq \frac{3}{2}d - 3$.

This is the situation considered by Severi: moving Z in a non empty open set $U \subset H_\delta$, and fixing d, g in the previous range, one can finally construct a unirational variety

$$\mathbb{P} := \bigcup_{Z \in U} |\mathcal{I}^2_Z(d)|$$

dominating $\mathcal{M}_g$. Still a subtlety is missed: the construction implicitly relies on a positive answer to the following question. Let Z be general in H_δ and let $h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - 3\delta \geq 1$, so that $|\mathcal{I}^2_Z(d)|$ is not empty:

Question 2.17 *Is a general* $\Gamma \in |\mathcal{I}^2_Z(d)|$ *integral and nodal of genus g?*

A simple counterexample actually exists. Let $\delta = 9$, $d = 6$ and Z general. Then $|\mathcal{I}^2_Z(6)|$ consists of a unique element Γ and $\Gamma = 2E$, where E is the unique plane cubic containing Z. Fortunately this is the unique exception, as is shown by

Arbarello and Sernesi in [AS]. This completes the description of the proof of the unirationality of $\mathcal{M}_g$, $g \leq 10$, via nodal plane curves.

For $g \geq 11$ one cannot go further with scrolls $\mathbb{P}$ in $\overline{\mathcal{V}}_{d,g}$ whose general element is a nodal curve. This remark is also proven by E. Sernesi (Unpublished note, 2010).

Theorem 2.18 *In $\overline{\mathcal{V}}_{d,g}$ there is no scroll $\mathbb{P}$ as above such that $m : \mathbb{P} \to \mathcal{M}_g$ is dominant and $g \geq 11$.*

Proof Assume that $\mathbb{P}$ exists. Let $\Gamma \in \mathbb{P}_t \subset \mathbb{P}$ be general and let $\sigma : S \to \mathbf{P}^2$ be the blowing up of Z_t. Since $\dim \mathbb{P}_t \geq 1$ the strict transform C of Γ is a curve of genus g such that $\dim |C| \geq 1$. Since Γ is nodal it follows that $C^2 \geq d^2 - 4\delta = -d^2 + 6d - 4 + 4g \geq 0$. Since C has general moduli we have also $g \leq \frac{3}{2}d - 3$. This implies that $d \leq 10$ and $g \leq 12$. The cases $g = 10, 11, 12$ can be excluded by an ad hoc analysis. □

Remark 2.19 Apparently, an intrinsic limit of the results we have outlined is the use of nodal plane curves. As remarked above $\overline{\mathcal{V}}_{d,g}$ is not ruled by linear spaces if it dominates $\mathcal{M}_g$ and $g \geq 11$. Equivalently let $C \subset S$ be a smooth, integral curve of genus $g \geq 11$ with general moduli in a rational surface S, then $\dim |C| = 0$. Of course this does not exclude the uniruledness of $\mathcal{V}_{d,g}$ for other reasons. This is for instance the case for $d = 10$ and $g = 11, 12$.

Leaving nodal plane curves, Beniamino Segre made a thorough attempt to construct scrolls $\mathbb{P} \subset |\mathcal{O}_{\mathbf{P}^2}(d)|$ such that

- *a general $\Gamma \in \mathbb{P}$ has genus g and ordinary singularities*,
- *Sing Γ is a set of points of $\mathbf{P}^2$ in general position*,
- *$m : \mathbb{P} \to \mathcal{M}_g$ is dominant.*

See [Se1]. This search gave negative answers:

Theorem 2.20 (B. Segre) *There is no scroll $\mathbb{P}$ for $g \geq 37$.*

A possible extension to $g \geq 11$ is also suggested by Segre. Let $\Gamma \in \mathbb{P}_t \subset \mathbb{P}$ and let $p_1 \dots p_s$ be the base points of $\mathbb{P}_t$, assumed to be general in $\mathbf{P}^2$. Consider the zero dimensional subscheme Z supported on them and locally defined at p_i by $\mathcal{I}_{p_i}^{m_i}$, where $\mathcal{I}_{p_i}$ is the ideal sheaf of p_i and m_i its multiplicity in Γ. We say that the linear system $|\mathcal{I}_Z(d)|$ is *regular* if it is not empty and $h^1(\mathcal{I}_Z(d)) = 0$. Segre shows that:

Theorem 2.21 *No scroll $\mathbb{P}$ exists for $g \geq 11$ if some $|\mathcal{I}_Z(d)|$ is regular.*

He says that the hypothesis regularity of $|\mathcal{I}_Z(d)|$ should follow from an unproved claim of intuitive evidence. This is probably the remote origin of Segre's conjecture, formulated much later in 1961, and the origin to many related conjectures:

Conjecture 2.22 If there exists a reduced curve $\Gamma \in |\mathcal{I}_Z(d)|$ then the linear system $|\mathcal{I}_Z(d)|$ is regular.

See [Se2] and [Ci1].

2.4 Rational Curves on $\overline{\mathcal{M}_g}$

The epilogue of the history we have described is well known to all algebraic geometers: Severi's conjecture was, somehow surprisingly, disproved. In 1982 Harris and Mumford proved that $\mathcal{M}_g$ is of general type as soon as g is odd and $g > 23$ [HM]. In 1984 Harris proved the same result in even genus [H1]. It was also proved that $\mathcal{M}_{23}$ has Kodaira dimension ≥ 0. The present updated picture, for every genus g, is as follows:

1. *$\mathcal{M}_g$ is rational for $g \leq 6$.*
2. *$\mathcal{M}_g$ is unirational for $g \leq 14$.*
3. *$\mathcal{M}_{15}$ is rationally connected, $\mathcal{M}_{16}$ is uniruled.*
4. *kod$(\mathcal{M}_g)$ is unknown for $g = 17, \dots, 21$.*
5. *$\mathcal{M}_g$ is of general type for $g = 22$ and $g \geq 24$.*
6. *$\mathcal{M}_{23}$ has Kodaira dimension ≥ 2.*

See [HMo1] or [Ve] for more details on several different contributions. It is time to quit the world of curves on rational surfaces and to discuss more in general, as far as the value of g makes it possible, the next

Question 2.23 *When does a general point of $\mathcal{M}_g$ lie in a rational curve?*

This is of course equivalent to ask wether $\mathcal{M}_g$ is uniruled. It is very easy to realize that this is also equivalent to discuss the next

Theme I *When does a general curve C embed in a smooth integral surface S so that the moduli map $m : |\mathcal{O}_S(C)| \to \mathcal{M}_g$ is not constant?*

The discussion made in the previous section also concerns the moduli of pairs (C, L) such that $L \in W^2_d(C)$, where d and the genus g of C are fixed. The coarse moduli space of pairs (C, L) such that $L \in W^r_d(C)$ will be

$$\mathcal{W}^r_{d,g}$$

and we will say that these spaces are the universal Brill-Noether loci. Passing from $\mathcal{M}_g$ to $\mathcal{W}^r_{d,g}$ a second theme is then natural here:

Theme II *Discuss the uniruledness problem for $\mathcal{W}^r_{d,g}$.*

Notice that Severi's method to prove the unirationality of $\mathcal{M}_g$, $g \leq 10$, immediately implies the unirationality of $\mathcal{W}^2_{d,g}$ in the same range, that is for $d \leq 9$ and $g \leq \frac{3}{2}d - 3$ On the other hand the minimal degree of a map $f : C \to \mathbf{P}^1$ is $k = [\frac{g+3}{2}]$ for a general curve C of genus g. Of special interest for these notes will be the case of $\mathcal{W}^1_{k,g}$.

We close this section by some concrete tests and examples on these themes, discussing curves with general moduli moving on non rational surfaces. This is going to involve K3 surfaces but also elliptic surfaces and some canonical surfaces in $\mathbf{P}^{g-1}$. As a byproduct in genus 10, we retrieve by a different method a counterexample to the slope conjecture given in [FP]. We use pencils of curves on some elliptic surfaces studied in [FV5].

Preliminarily we recall some basic facts on the compactification of $\mathcal{M}_g$ by the moduli of stable curves $\overline{\mathcal{M}}_g$, see e.g. [ACG, Chaps. 12 and 13], [HMo, F].

$\overline{\mathcal{M}}_g$ is an integral projective variety with canonical singularities. Denoting by $[D]$ the class of the divisor D, the Picard group of $\overline{\mathcal{M}}_g$ is generated over $\mathbb{Q}$ by the following divisorial classes:

- $\lambda = [det\ \Lambda]$, Λ being the Hodge bundle with fibre $H^0(\omega_C)$ at $[C]$.
- $\delta_0 = [\Delta_0]$. For a general $[C] \in \Delta_0$ C is 1-nodal and integral.
- $\delta_i = [\Delta_i]$. For a general $[C] \in \Delta_0$ C is 1-nodal and $C = C_i \cup C_{g-i}$ where C_i and C_{g-i} are smooth, integral of genus i and $g-i$ respectively.

For the canonical class we have

$$[K_{\overline{\mathcal{M}}_g}] = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[\frac{g}{2}]} := k_g.$$

Putting as usual $\delta := \sum_{i=0\ldots[\frac{g}{2}]} \delta_i$ we have for the canonical class

$$k_g = 13\lambda - 2\delta - \delta_1.$$

More in general we can consider divisors D such that

$$[D] = a\lambda - b\delta - \sum c_i\delta_i,$$

with $a, b > 0$ and $c_i \geq 0$, and define the slope of any divisor as follows [HMo].

Definition 2.24 Let D be any divisor: if $[D] = a\lambda - b\delta - \sum c_i\delta_i$ as above, then the slope of D is the number $s(D) := \frac{a}{b}$. If not we put $s(D) := \infty$.

Definition 2.25 The slope of $\overline{\mathcal{M}}_g$ is

$$s_g := \inf\{s(E),\ [E] \in NS(\overline{\mathcal{M}}_g) \otimes \mathbb{R}\ E \text{ effective divisor}\}.$$

As we can see one has $s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}$ for the slope of $K_{\overline{\mathcal{M}}_g}$.

Can we deduce something about the effectiveness of $K_{\overline{\mathcal{M}}_g}$? It is natural to answer quoting the theorem of Boucksom et al. [BDPP]. The theorem says that an integral projective variety with canonical singularities is non uniruled iff its canonical divisor is limit of effective divisors (pseudoeffective). A uniruledness criterion for $\overline{\mathcal{M}}_g$ is therefore:

$$s_g > s(k_g) = \frac{13}{2}.$$

How to apply this criterion? A typical way involves the study of the irreducible flat families of curves in $\overline{\mathcal{M}}_g$ having the property that a general point of $\overline{\mathcal{M}}_g$ belongs to some members of the family. We say that a family like this is a *family of curves sweeping $\overline{\mathcal{M}}_g$ or a sweeping family*. Let

$$m : B \to \overline{\mathcal{M}}_g$$

be a non constant morphism where B is an integral curve. Up to base change and stable reduction we can assume that B is smooth of genus p and endowed with a flat family $f : S \to B$ of stable curves such that

$$m(b) = \text{the moduli point of } f^*(b).$$

Consider any effective divisor $E \subset \overline{\mathcal{M}}_g$ and a non constant morphism

$$m : B \to \overline{\mathcal{M}}_g$$

such that $m(B)$ is a sweeping curve. Then we can assume $m(B) \not\subset E$, so that $\deg m^*E \geq 0$. The next lemma is well known and summarizes some properties of the numerical characters of a fibration of a surface onto a curve.

Lemma 2.26

- *$deg\, m^*E \geq 0$.*
- *$s(E) \geq \frac{m^*\delta}{m^*\lambda}$.*
- $\deg\ m^*\delta = c_2(S) + (2 - 2p)(2g - 2)$
- $\deg m^*\lambda = \deg\ \omega_{S/B}$.

The next theorem is an elementary but crucial corollary:

Theorem 2.27 *Let $m : B \to \overline{\mathcal{M}}_g$ be a morphism such that $m(B)$ moves in a sweeping family of curves. Then $\frac{\deg m^*\delta}{\deg m^*\lambda} > \frac{13}{2}$ implies that $\overline{\mathcal{M}}_g$ is uniruled.*

In the next examples we only deal with the case $B = \mathbf{P}^1$ that is with rational curves $m(B)$ in the moduli space $\overline{\mathcal{M}}_g$.

Here is a recipe to construct examples in this case: in a regular surface S construct a smooth integral curve $C \subset S$, having genus g and such that $h^0(\mathcal{O}_C(C)) \geq 1$. Then the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

implies $\dim |C| \geq 1$. Take, possibly, a Lefschetz pencil $B \subset |C|$. Then the map $m : B \to \overline{\mathcal{M}}_g$ provides a rational curve $m(B)$ in $\overline{\mathcal{M}}_g$. Let $\sigma : S' \to S$ be the blow up of the base locus of B, it is standard to compute that

$$deg\ m^*\delta = c_2(S') + 4g - 4\ ,\ deg\ m^*\lambda = \chi(\mathcal{O}_S) + g - 1.$$

*Example 2.28 (*K3 SURFACES*)* This is a very important case, to be discussed later as well. We will consider pairs (S, L) where S is a K3 surface and L a big and nef line bundle such that $c_1(L)^2 = 2g - 2$. If $g \geq 3$ and (S, L) is general L is very ample. Then L defines an embedding $S \subset \mathbf{P}^g$ with canonical curves of genus g as hyperplane sections. Let B be a general pencil of hyperplane sections of S, applying the previous formulae to B we obtain:

$$\frac{deg\ m^*\delta}{deg\ m^*\lambda} = 6 + \frac{12}{g+1}.$$

This equality fits very well with the first results of Mumford-Harris on $\mathcal{M}_g$, which say that $\mathcal{M}_g$ is of general type for $g > 23$. Indeed we have

$$6 + \frac{12}{g+1} > \frac{13}{2}$$

exactly for $g > 23$. Though $m(B)$ does not move in a sweeping family, unless $g \leq 11$ and $g \neq 10$, these numbers lend some plausibility to the following

Conjecture 2.29 The slope of $\overline{\mathcal{M}}_g$ is $s_g = 6 + \frac{12}{g+1}$

See [HMo1] for further precision on this statement, which is known as *slope conjecture*. It was however disproved by Farkas and Popa in [FP] for infinite values of g, starting from the very interesting case $g = 10$.

Example 2.30 (Elliptic Surfaces) Here we give another proof of the counterexample to the slope conjecture for $g = 10$. We use elliptic surfaces with $p_g = 2$ and $q = 0$, endowed with a very ample linear system of curves C of genus 10 and such that $C^2 = 12$.

Actually we describe and apply the results in [FV5] on curves of even genus g on some classes of elliptic surfaces. Let $g = 2k$, in [FV5] an integral family $\mathcal{S}_k$ of projective surfaces

$$S \subset \mathbf{P}^k$$

is constructed. They are elliptic surfaces with $p_g = 0$ and $q = 0$. We assume $k \geq 5$ to have that a general S is smooth. Let C be a smooth hyperplane section of S. The notable properties of this construction are as follows:

- *C has genus $g = 2k$,*
- *$|K_S|$ is a pencil of elliptic curves of degree $k + 1$,*
- $M := \mathcal{O}_C(K_S) \in W^1_{k+1}(C)$,
- $\mathcal{O}_C(1) \cong \omega_C(-M)$.

Notice that $W^1_{k+1}(C)$ is finite if C is general and no pencil of divisors of degree $\leq k$ exists on C. Furthermore let S be general in $\mathcal{S}_k$, then:

- S is projectively normal,
- C has general moduli for $g \leq 12$.

Coming to genus 10, the family of curves of this genus which can be embedded in a K3 surface play a special role for $\overline{\mathcal{M}}_{10}$. As is well known the locus of these curves is an irreducible divisor $D_{K3} \subset \overline{\mathcal{M}}_{10}$. Let us sketch a proof that its slope is 7 so contradicting the slope conjecture $s_{10} = 6 + \frac{12}{11}$.

Putting $k = 5$ fix a general pencil $B \subset |C|$ on a general elliptic surface $S \subset \mathbf{P}^5$ as above. One can show that B is a Lefschetz pencil, then every $C \in B$ is integral and nodal with at most one node. Since S is projectively normal, the multiplication map

$$\mu : Sym^2 H^0(\mathcal{O}_C(1)) \to H^0(\mathcal{O}_C(2))$$

is of maximal rank, in our case it is an isomorphism, for every $C \in B$. Equivalently C is not embedded in a K3 surface of degree 6 of the hyperplane $< C >= \mathbf{P}^4$. On the other hand $\mathcal{O}_C(1)$ belongs to $W^4_{12}(C)$. Then the results of Voisin in [V], cfr. 3.2 and 4.13 (b), imply that the multiplication map $\mu_A : Sym^2 H^0(A) \to H^0(A^{\otimes 2})$ is isomorphic for any $M \in W^4_{12}(C)$ and hence that no $C \in B$ embeds in a K3 surface. This implies that

Lemma 2.31 $m^* D_{K3} = 0$.

One can also show that $[D_{K3}] = a\lambda - b\delta - \sum c_i \delta_i$, where $i > 0, a > 0$ and $b, c_i \geq 0$. Since B is a Lefschetz pencil, it follows that $m^*\delta_i = 0$ for $i > 0$. Moreover, by the previous formulae, we compute

$$\deg\ m^*\delta = 84\ ,\ \deg\ m^*\lambda = \chi(\mathcal{O}_S) + (g-1) = 12.$$

Hence, by Lemma 2.31, it follows $\deg m^* D_{K3} = 12a - 84b = 0$ so that $s(D_{K3}) = \frac{a}{b} = 7$. This contradicts the slope conjecture.

Example 2.32 (Canonical Surfaces) Let $d = 10$ and $g = 11, 12$ we give a hint to prove that $\mathcal{W}^2_{10,g}$ is uniruled.

Assume that C of genus g has general moduli and that $L \in W^2_{10}(C)$. Consider the embedding $C \subset \mathbf{P}^2 \times \mathbf{P}^n$ defined by the product of the maps associated to L and $\omega_C(-L)$, $n = g - 9$.

Let $g = 11$: since $n = 2$, C is embedded in $S \subset \mathbf{P}^2 \times \mathbf{P}^2$, S is a smooth complete intersection S of two hypersurfaces of bidegree $(2, 2)$. Then S is a regular canonical surface in the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$, and $\mathcal{O}_C(1, 1)$ is ω_C. Hence $\mathcal{O}_C(C)$ is trivial by adjunction formula and $\dim |C| = 1$.

Let $D \in |C|$ then D is endowed with $L_D := \mathcal{O}_D(1, 0) \in W^2_{10}(D)$. The image of the corresponding moduli map $m : |C| \to \mathcal{W}^2_{10,1}$ is a rational curve passing through a general point of $\mathcal{W}^2_{10,11}$.

Let $g = 12$: since $n = 3$, C is embedded in $S \subset \mathbf{P}^2 \times \mathbf{P}^3$, S is a complete intersection of three hypersurfaces of bidegrees (1,2), (1,2), (2,1). Then the argument of the proof is the same.

3 $\mathcal{R}_g$ in Genus at Most 6

3.1 Prym Curves and Their Moduli

A smooth *Prym curve* of genus g is a pair (C, η) such that C is a smooth, connected curve of genus g and η is a non zero 2-torsion element of $Pic^0(C)$.

It is useful to recall the following characterizations of a pair (C, η) and to fix consequently the notation. Keeping C fixed let us consider the sets:

$T :=$ {non trivial line bundles $\eta \in Pic(C) \mid \eta^{\otimes 2} \cong \mathcal{O}_C,$ },
$E :=$ {non split étale double coverings $\pi : \tilde{C} \to C$}.
$I :=$ {fixed point free involutions i on a connected curve $\tilde{C} | C \cong \tilde{C}/ < i >$}.

Then the next property is standard and well known.

Theorem 3.1 *The sets T, I, E are naturally bijective.*

Indeed let $\eta \in T$, then there exists a unique isomorphism $\beta : \mathcal{O}_C \to \eta^{\otimes 2}$ modulo $\mathbf{C}^*$. From it one uniquely defines the projective curve

$$\tilde{C} := \{(1 : v) \in \mathbb{P}(\mathcal{O}_C \oplus \eta)_x \;/\; v \otimes v = \beta_x(1),\; x \in C\} \subset \mathbf{P}(\mathcal{O}_C \oplus \eta).$$

$\tilde{C}$ is a smooth, integral curve of genus $2g - 1$. It is endowed with:

- the fixed point free involution $i : \tilde{C} \to \tilde{C}$ such that $i(u : v) = (u : -v)$,
- the étale 2:1 cover $\pi : \tilde{C} \to C$ induced by the projection $\mathbf{P}(\mathcal{O}_C \oplus \eta) \to C$.

The assignments $\eta \to (\tilde{C}, i)$ and $\eta \to \pi$ define the required bijections $T \leftrightarrow I$ and $T \leftrightarrow E$. π is the quotient map of i. We omit more details.

Let (C, η) be a Prym curve, throughout all this exposition we will keep the notation $\pi : \tilde{C} \to C$ and $i : \tilde{C} \to \tilde{C}$ for the maps constructed as above from (C, η). We can now begin with the following:

Definition 3.2 The Prym moduli space of genus g is the moduli space of smooth Prym curves of genus g. It will be denoted as $\mathcal{R}_g$.

As it is well known $\mathcal{R}_g$ is an integral quasi projective variety of dimension $3g-3$. Actually the forgetful map $f : \mathcal{R}_g \to \mathcal{M}_g$ is a finite morphism of degree $2^{2g} - 1$, with fibre $T \setminus \{\mathcal{O}_C\}$ over the moduli point of C. The notion of smooth Prym curve needs to be extended, in order to construct suitable compactifications of $\mathcal{R}_g$. The moduli space of quasi stable Prym curves provides a useful compactification, that we now describe. See [FL].

A component E of a semistable curve C is said to be *exceptional* if E is a copy of $\mathbf{P}^1$ and $E \cap \overline{C - E}$ is a set of two points.

Definition 3.3 A semistable curve C is quasi stable if its exceptional components are two by two disjoint.

We recall that a vector bundle on C has degree d if d is the sum of the degrees of its restrictions to the irreducible components of C.

Definition 3.4 (C, η, β) is a quasi stable Prym curve of genus g if:

- C is quasi stable of genus g and η is a line bundle on C of degree 0,
- $\beta : \mathcal{O}_C \to \eta^{\otimes 2}$ is a morphism of sheaves,
- let D be the union of the exceptional components of C, then β is an isomorphism on $C - D$.

For short we will say that (C, η, β) is a Prym curve of genus g. The moduli space of these triples is denoted as $\overline{\mathcal{R}}_g$. This is a projective normal variety with finite quotient singularities and $\mathcal{R}_g$ is a dense open subset of it.

$\overline{\mathcal{R}}_g$ has good properties in order to study the Kodaira dimension, as it is done by Farkas and Ludwig in [FL]. They show that any resolution of singularities $X \to \overline{\mathcal{R}}_g$ induces an isomorphism between the spaces of pluricanonical forms $H^0(\omega_X^{\otimes m})$ and $H^0(\omega_{\overline{\mathcal{R}}_g}^{\otimes m})$, $m \geq 1$. Then the key point is the study of the forgetful map

$$f : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$$

sending $[C, \eta, \beta]$ to $[st(C)]$, where $st(C)$ denotes the stable model of C. It turns out that f is finite and ramifies, with simple ramification, precisely on

$$D_0^{ram} := \{[C, \eta, \beta] \ / \ C \ \textit{contains exceptional components}\},$$

which is an irreducible divisor. In particular f offers a useful description of the canonical class of $\overline{\mathcal{R}}_g$ in terms of D_0^{ram} and of the pull back of the canonical class of $\overline{\mathcal{M}}_g$. An outcome of this approach is the following:

Theorem 3.5 *$\overline{\mathcal{R}}_g$ is of general type for $g \geq 14$ and $g \neq 15$. The Kodaira dimension of $\overline{\mathcal{R}}_{15}$ is bigger or equal than* 1.

Though needed later, let us introduce now the picture of the boundary of $\overline{\mathcal{R}}_g$. Its irreducible components are divisors which are obtained from the boundary divisors $\Delta_i, \ i = 0 \ldots [\frac{g}{2}]$, of $\overline{\mathcal{M}}_g$ as follows. Let $\overline{C} := st(C)$, consider the standard exact sequence

$$0 \to \mathbf{C}^{*k} \to Pic^0 \, \overline{C} \stackrel{\nu^*}{\to} Pic^0 \, N \to 0$$

where $\nu : N \to \overline{C}$ is the normalization map and $k = |Sing \, \overline{C}|$. We have:

$$f^*(\Delta_0) = D_0' + D_0'' + 2D_0^{ram}$$

so that D_0' and D_0'' are the following irreducible divisors

- $D_0' := \{[C, \eta, \beta] \in \overline{\mathcal{R}}_g \ / \ [\overline{C}] \in \Delta_0 \ , \ \nu^*\eta \ \textit{is non trivial}\}$
- $D_0'' := \{[C, \eta, \beta] \in \overline{\mathcal{R}}_g \ / \ [\overline{C}] \in \Delta_0 \ , \ \nu^*\eta \ \textit{is trivial}\}$

Remark 3.6 Let $p := [C, \eta, \beta] \in f^*(\Delta_0)$ be a general point then $\overline{C}$ is integral. Let $\pi : \tilde{C} \to C$ be the morphism defined by η. If $p \in D_0'$ then $C = \overline{C}$ and π is étale. If $p \in D_0''$ then $C = \overline{C}$ and π is a Wirtinger covering, cfr. [FL]. If $p \in D_0^{ram}$ then π is obtained from a $2:1$ cover of $\overline{C}$ branched on the tangent directions of the nodes.

Moreover we have

$$f^*(\Delta_i) = D_i + D_{g-i} + D_{i:g-i}$$

for $i = 1 \leq i \leq [\frac{g}{2}]$. Here a general point $p = [C, \eta, \beta] \in f^*(\Delta_i)$ is such that

$$C = C_i \cup C_{g-i},$$

where C_i and C_{g-i} are smooth, integral curve of genus i and $g-i$. By definition $p \in D_{i:g-i}$ if η is non trivial on both C_i and C_{g-i}. Moreover let $x = i, g-i$ then $p \in D_x$ if η is non trivial on C_x and trivial on C_{g-x}. As usual we will denote the classes of the previous divisors as follows:

$$[\Delta_0'] := \delta_0' \ , \ [\Delta_0''] := \delta_0'' \ , \ [\Delta_0^{ram}] := \delta_0^{ram} \ , \ [D_i] = \delta_i \ , \ [D_{i:g-i}] := \delta_{i:g-i}.$$

Considering f it turns out that the canonical class of $\overline{\mathcal{R}}_g$ is

$$K_{\overline{\mathcal{R}}_g} \equiv 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{ram} - 2 \sum_{i=1\dots\frac{g}{2}} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}).$$

3.2 Unirational Approaches to $\mathcal{R}_g$

Rationality problems are connected to Prym curves, their moduli and related topics by very important links and this fact is also well visible in the historical evolution of the two fields. It is the moment for opening some perspectives about.

The most relevant link between rationality problems and Prym curves appears to be the notion of Prym variety $P(C, \eta)$ associated to a smooth Prym curve (C, η) of genus g. $P(C, \eta)$ is a $g-1$-dimensional principally polarized abelian variety we more precisely define later.

Prym varieties appear as intermediate Jacobians of several unirational threefolds. Notably we have among them any smooth cubic threefold X. They play a crucial role to prove that X is not rational, hence to produce counterexamples to Lüroth problem. This was a strong motivation to modern studies on Prym varieties and Prym curves.

Secondly the assignment $(C, \eta) \longrightarrow P(C, \eta)$ defines the *Prym map*

$$P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}.$$

The Prym map is a fundamental tool, essentially the unique one, to prove the unirationality of $\mathcal{A}_{g-1}$ for $g \le 6$. In the same range it is also useful for understanding more of the birational geometry of $\mathcal{A}_{g-1}$. For instance one can study the slope of a suitable compactification via the parametrization offered by the Prym map. This is possible in low genus thanks to a long history of results on the unirationality of $\mathcal{R}_g$ to be outlined here.

We highlight some old and some new approaches to the problem. These are often induced by analogous questions, or themes, we considered for $\mathcal{M}_g$ in the previous section. Let us recall the following ones:

Theme I *When does a Severi variety $\mathcal{V}_{d,g}$ dominate $\mathcal{M}_g$ and contain an open subset covered by rational curves with non constant moduli?*

Theme II *When does a curve C with general moduli embed in a smooth surface S so that the moduli map $m : |\mathcal{O}_S(C)| \to \mathcal{M}_g$ is not constant?*

We introduce some versions for Prym curves of these themes.

Let $p : P \to \mathbf{P}^2$ be a $\mathbf{P}^2$-bundle endowed with a very ample tautological bundle $\mathcal{O}_P(1)$. Then a general $Q \in |\mathcal{O}_P(2)|$ is a smooth conic bundle

$$p/Q : \ Q \to \mathbf{P}^2.$$

Let $\Gamma \subset \mathbf{P}^2$ be the discriminant curve of Q. It is well known that Γ is smooth for a general smooth Q and endowed with a non split étale double cover $\pi_\Gamma : \tilde{\Gamma} \to \Gamma$. In particular π_Γ is defined by a non trivial $\eta_\Gamma \in Pic^0(\Gamma)$ such that $\eta_\Gamma^{\otimes 2} \cong \mathcal{O}_\Gamma$. Hence p/Q defines the Prym curve (Γ, η_Γ). The same happens choosing Q general in the Severi variety of nodal conic bundles

$$\mathcal{V}_{P,\delta} := \{Q \in |\mathcal{O}_P(2)| \ \mid \textit{Sing } Q \textit{ consists of } \delta \textit{ ordinary double points}\}.$$

In this case Γ is integral with δ nodes. Let d be the degree of Γ and C its normalization. Then p/Q induces a non split étale double cover $\pi : \tilde{C} \to C$, hence a smooth Prym curve (C, η) of genus $g = \frac{1}{2}(d-1)(d-2) - \delta$. The assignment $Q \to (C, \eta)$ defines the moduli map

$$m : \mathcal{V}_{P,\delta} \to \mathcal{R}_g.$$

Theme I [Nodal conic bundles] *When is m dominant and does a general curve $\Gamma \in \mathcal{V}_{P,\delta}$ move in a rational family with non constant moduli?*

Let (C, η) be a Prym curve of genus g with general moduli, going back to surfaces the second theme is:

Theme II [Surfaces] *Does it exist an embedding $C \subset S$ and $E \in Pic\ S$ so that S is a smooth surface, $\eta \cong \mathcal{O}_S(E)$ and $m :| C |\to \mathcal{R}_g$ is not constant?*

What about unirationality and rationality of $\mathcal{R}_g$ in low genus? Answering this question we can revisit, by the way, some nice classical constructions.

The unirationality of $\mathcal{R}_g$ is known for $g \leq 7$. In genus 7 this is a recent result due to Farkas and the author [FV6].

In genus 6 the unirationality was independently proven by Mori and Mukai in [MM], by Donagi in [Do1] and in [Ve2]. Donagi's proof uses nets of quadrics of $\mathbf{P}^6$ whose discriminant curve splits in the union of a line and a 4-nodal sextic. The other two proofs rely on Enriques surfaces.

The case of genus 5 was more recently proved, see [ILS] and [Ve3]. The work of Clemens, on rational parametrizations of $\mathcal{A}_p$, $p \leq 4$, via nodal quartic double solids, is strictly related to these results [Cl].

A very recent quick proof of the unirationality of $\mathcal{R}_g$, $g \leq 6$, is given in [FV4] and presented here, see Sect. 3.4. It uses linear systems of nodal conic bundles in the spirit of Theme I.

3.3 Rationality Constructions

The rationality of $\mathcal{R}_g$ is known for $g \leq 4$. For $g = 2, 3$ it follows from the results of various authors: Catanese [Ca], Katsylo [K2] and Dolgachev [D1], starting from the first published result due to Katsylo.

For $\mathcal{R}_4$ the rationality result is due to Catanese. This is perfectly in the spirit of theme II and we revisit it. For $\mathcal{R}_4$ we are back, as in the case of $\mathcal{M}_6$, to a linear system of nodal plane sextics. Let $(x_1 : x_2 : x_3)$ be projective coordinates on $\mathbf{P}^2$ and let $u = x_1 + x_2 + x_3$. The linear system we want to consider can be written very explicitly as follows:

$$(x_1x_2x_3u)q + b_1(x_2x_3u)^2 + b_2(x_1x_3u)^2 + b_3(x_1x_2u)^2 + b_4(x_1x_2x_3)^2 = 0,$$

where $q := \sum_{1 \leq i \leq j \leq 3} a_{ij}x_ix_j$. Clearly such a linear system is 9-dimensional. We denote it as $\mathbf{P}^9$ and the coefficients $(a : b) := (a_{ij} : b_1 : \cdots : b_4)$ are projective coordinates on $\mathbf{P}^9$. We point out that $\mathbf{P}^9$ is precisely the linear system of sextics which are singular along the nodes of the *quadrilateral*

$$T = \{x_1x_2x_3u = 0\}.$$

It is easy to prove that a general $\Gamma \in \mathbf{P}^9$ is an integral nodal curve. In particular let $\nu : C \to \Gamma$ be the normalization map, then C has genus 4. Now we consider the effective divisors $h \in |\nu^*\mathcal{O}_\Gamma(1)|$ and $n = \nu^* Sing\ \Gamma$. Note that n is just the sum of the points over the six nodes of Γ.

Lemma 3.7 *Let $\eta := \mathcal{O}_C(n - 2h)$ then:*

- *η is non trivial*
- *$\eta^{\otimes 2} \cong \mathcal{O}_C$.*
- *$\mathcal{O}_C(h) \cong \omega_C \otimes \eta$.*

Proof One has $2n = \nu^*T \sim 4h$ so that $\eta^{\otimes 2} \cong \mathcal{O}_C(4h - 2n) = \mathcal{O}_C$. Assume η itself is trivial, then $2h \sim n$ and hence $3h - n \sim h$. Since $3h - n$ is the canonical class K_C it follows that Γ is the image of a map defined by a net $N \subset |K_C|$. This implies that a line section of Γ pulls back by ν to a canonical divisor and, by adjunction theory for plane curves, that the six nodes of Γ are on a conic. This is a contradiction because $Sing\ \Gamma = Sing\ T$. Finally $K_C \sim 3h - n$ and $\eta \cong \mathcal{O}_C(n - 2h) \Rightarrow \mathcal{O}_C(h) \cong \omega_C \otimes \eta$.□

It follows that (C, η) is a Prym curve and this defines a moduli map

$$m : \mathbf{P}^9 \to \mathcal{R}_4$$

as usual. Conversely, starting from (C, η), we can reconstruct Γ modulo projective automorphisms of $\mathbf{P}^2$. Indeed $\dim |\omega_C \otimes \eta| = 2$ and the image of the map defined by $\omega_C \otimes \eta$ is projectively equivalent to Γ. This implies that Γ is uniquely reconstructed from (C, η), modulo the group of projective automorphisms which are leaving invariant the linear system $\mathbf{P}^9$. This is precisely the stabilizer of the quadrilateral T, hence it is the symmetric group $\mathfrak{s}_4$ and m has degree 24. Since $\mathbf{P}^9$ and $\mathcal{R}_4$ are integral of the same dimension, it follows that m is dominant. Therefore we can conclude that

$$\mathbf{P}^9/\mathfrak{s}_4 \cong \mathcal{R}_4.$$

Analyzing the action of $\mathfrak{s}_4$ it follows that the quotient $\mathbf{P}^9/\mathfrak{s}_4$ is rational [Ca]. This shows that

Theorem 3.8 *The Prym moduli space $\mathcal{R}_4$ is rational.*

Some special Del Pezzo surfaces arise as an interesting complement of the geometry of $\mathcal{R}_4$. Let $\mathcal{I}_{Sing\ T}$ be the ideal sheaf of the six singular points of T. Then the linear system of plane cubics $|\mathcal{I}_{Sing\ T}(3)|$ defines a generically injective rational map $f : \mathbf{P}^2 \to \mathbf{P}^3$ whose image $\overline{S} := f(\mathbf{P}^2)$ is a 4-nodal cubic surface. It is worth mentioning some more geometry related to the Prym curve (C, η) and to the sextic Γ. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{f'} & \tilde{S} \\ \pi \downarrow & & \downarrow \overline{\pi} \\ \mathbf{P}^2 & \xrightarrow{f} & \overline{S} \end{array}$$

which is described as follows:

- f contracts the 4 lines of T to $Sing\ \overline{S}$ and blows up the 6 points of $Sing\ T$ to 6 lines in $\overline{S}$. These are the edges of the tetrahedron $Sing\ \overline{S}$.
- $\pi : \tilde{S}' \to \mathbf{P}^2$ is the finite 2:1 cover branched on T and $\tilde{S}'$ is a singular Del Pezzo surface of degree 2 with six nodes, $Sing\ \tilde{S}' = \pi^{-1}(Sing\ T)$.

- After blowing up $Sing\ \tilde{S}'$, the strict transforms of the irreducible components of $\pi^{-1}(T)$ become -1 lines.
- f' is the contraction of these lines and $\tilde{S}$ is a smooth Del Pezzo surface of degree 6. We leave the completion of many details to the reader.

Also, we just mention that:

- the strict transform of Γ by f is the canonical model $\overline{C}$ of C.
- $\tilde{C} := \overline{\pi}^{-1}(\overline{C})$ is the étale double cover of C defined by η.
- $\tilde{\Gamma} := \pi^{-1}(\Gamma)$ is birational to $\tilde{C}$.

3.4 Nodal Conic Bundles for g ≤ 6

Following the approach suggested in Theme I we construct here rational families of nodal conic bundles dominating $\mathcal{R}_g$ for $g \leq 6$. This will offer a unique and quick method for proving all unirationality results in this range.

We fix a $\mathbb{P}^2$-bundle $p : \mathbb{P} \to S$ over a smooth rational surface S admitting a very ample tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$. As already remarked a general $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ is an integral threefold such that the projection

$$p/Q : Q \to S$$

is flat. This means in our situation that every fibre $Q_x := (p/Q)^*x$ is a conic in the plane $\mathbb{P}_x := p^*x$. We will assume that either Q is smooth or that $Sing\ Q$ consist of finitely many ordinary double points. Then it follows that the branch locus of p/Q is either empty or the curve

$$\Gamma := \{x \in S \ / \ rk\ Q_x \leq 2\}.$$

We assume that Γ is not empty and say that Γ is the *discriminant curve* of p/Q. The next lemma follows from [Be1] I.

Lemma 3.9 *Let $x \in S - p(Sing\ Q)$, then $x \in Sing\ \Gamma$ iff Q_x has rank one.*

As it is well known Γ is endowed with a finite double cover

$$\pi_\Gamma : \tilde{\Gamma} \to \Gamma,$$

where $\tilde{\Gamma}$ parametrizes the lines which are components of Q_x, $x \in \Gamma$. Let us fix from now on the next assumption, which is generically satisfied by a general $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ in all the cases to be considered:

- *Q_x has rank two at each point $x \in p(Sing\ Q)$.*

It is not difficult to see that $p(Sing\ Q) \subseteq Sing\ \Gamma$ and that Γ is nodal. Moreover, under the previous assumptions the next lemma follows.

Lemma 3.10 *Let $Sing\ \Gamma = p(Sing\ Q)$, then the morphism*

$$\pi_\Gamma : \tilde{\Gamma} \to \Gamma$$

is an étale double covering.

Let $Sing\ Q = Sing\ \Gamma$, then to give π_Γ is equivalent to give a line bundle η_Γ on Γ whose square is $\mathcal{O}_\Gamma$. It is well known that $\pi_{\Gamma *}\mathcal{O}_{\tilde{\Gamma}} \cong \mathcal{O}_\Gamma \oplus \eta_\Gamma$ and, moreover, π_Γ is uniquely reconstructed from η_Γ. In particular η_Γ is an element of $Pic^0_2\ \Gamma$. Let $\nu : C \to \Gamma$ be the normalization map, then we have the standard exact sequence of 2-torsion groups

$$0 \to (\mathbf{C}^*)^\delta_2 \to Pic^0_2\ \Gamma \xrightarrow{\nu^*} Pic^0_2\ C \to 0,$$

$\delta = |Sing\ \Gamma|$. It will be not restrictive to assume $\eta_\Gamma \neq \mathcal{O}_\Gamma$ so that π_Γ *is not split*. For a non trivial $\eta_\Gamma \in Ker\ \nu^*$ we say that π_Γ is a *Wirtinger covering*. Notice that, since $\nu^*\eta_\Gamma$ is trivial, π_Γ is Wirtinger or split iff the induced morphism $\pi : \tilde{C} \to C$ of the normalized curves is split.

We will say that $Q \in |\mathcal{O}_\mathbb{P}(2)|$ is a *conic bundle* if $Sing\ Q$ consists of finitely many ordinary double points and all the assumptions we made are satisfied. $\forall\ \delta \geq 0$ we consider the Severi varieties of nodal conic bundles

$$V_{\mathbb{P},\delta} = \{Q \in |\mathcal{O}_\mathbb{P}(2)| \ /\ Q \text{ as above and } |Sing\ Q| = \delta\}.$$

In view of our application in genus $g \leq 6$, it will be sufficient to consider the trivial projective bundle, so we put $\mathbb{P} := \mathbf{P}^2 \times \mathbf{P}^2$ and $\mathcal{O}_\mathbb{P}(1) := \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1)$. Fixing coordinates $(x, y) = (x_1 : x_2 : x_3) \times (y_1 : y_2 : y_3)$ on $\mathbf{P}^2 \times \mathbf{P}^2$, the equation of an element $Q \in |\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2,2)|$ is

$$\sum a_{ij}(x)y_iy_j = 0.$$

For Q general the discriminant curve Γ is a smooth sextic. Its equation is $det\ (a_{ij}) = 0$. The 2:1 cover $\pi_\Gamma : \tilde{\Gamma} \to \Gamma$ is étale and defined by a non trivial element $\eta_\Gamma \in Pic^0_2\ \Gamma$. To give η_Γ is equivalent to give the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-4)^3 \xrightarrow{A} \mathcal{O}_{\mathbf{P}^2}(-2)^3 \to \eta_\Gamma \to 0,$$

where $A = (a_{ij})$ is a symmetric matrix of quadratic forms, see [Be2]. The construction extends to nodal plane sextics Γ with δ nodes, see [FV4] for the details. We consider the most important case, namely the case $\delta = 4$. It will be enough to fix the four nodes of our conic bundle as follows:

$$(o_1, o_1) \dots (o_4, o_4) \in \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8.$$

Let O be the set of these points and $\mathcal{I}_O$ its ideal sheaf, we then have the linear system of conic bundles

$$\mathbf{P}^{15} := |\mathcal{I}_O^2(2,2)| \subset |\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)|.$$

The general $Q \in \mathbf{P}^{15}$ is a 4-nodal conic bundle satisfying our previous assumptions. Furthermore we have a natural map

$$m : \mathbf{P}^{15} \to \mathcal{R}_6$$

sending Q to $[C, \eta]$, where $\nu : C \to \Gamma$ is the normalization map and $\eta := \nu^* \eta_\Gamma$. This map is actually dominant. To see this take a general $[C, \eta]$ and any sextic model of C as a 4-nodal plane sextic Γ. Fixing any η_Γ such that $\nu^* \eta_\Gamma \cong \eta$ one can reconstruct as above a symmetric matrix of quadratic forms $A = (a_{ij})$ such that $det\ A = 0$ is the equation of Γ. Moreover A defines the conic bundle Q of equation $\sum a_{ij} y_i y_j = 0$. Finally, up to biregular automorphisms of $\mathbb{P}$, Q belongs to $|\mathcal{O}_{\mathbb{P}}(2)|$ and clearly $m(Q) = [C, \eta]$.

It is not difficult to extend this argument to the case $\delta \geq 5$ in order to construct a rational family of linear systems of δ-nodal conic bundles which dominates $\mathcal{R}_g$, $g \leq 5$. This concludes a very quick proof of the next

Theorem 3.11 *$\mathcal{R}_g$ is unirational for $g \leq 6$.*

Remark 3.12 A further use of families of nodal conic bundles, in higher genus and in some $\mathbf{P}^2$-bundle over a rational surface S, could be a priori not excluded. However this approach seems difficult and we are aware of a very small number of possible applications. As we will see, the unirationality of $\mathcal{R}_7$ is better reached via K3 surfaces.

Now we want to desingularize, so to say, the previous family of 4-nodal conic bundles in order to see more geometry of it. More precisely we want to pass, by suitable birational transformations, from $\mathbf{P}^2 \times \mathbf{P}^2$ to the $\mathbf{P}^2$-bundle $p : \mathbb{P} \to S$, where S is a smooth quintic Del Pezzo surface embedded in the Grassmannian $G(3,5)$ and $\mathbb{P}$ is the universal plane over it. We start from the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ and consider the linear projection

$$h : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4.$$

of center the set of four points O. Since the degree of the Segre embedding is six, then h is a rational dominant map of degree two. The map h induces a generically injective rational map

$$h_* : \mathbf{P}^2 \dashrightarrow S \subset G(3,5).$$

h_* defines a *congruence of planes* S of $\mathbf{P}^4$ that is a surface in the Grassmann variety $G(3,5)$. It is easy to check that S is a smooth quintic Del Pezzo surface and a linear section of $G(3,5)$. Let $\sigma : S \to \mathbf{P}^2$ be the blowing up of the set of four points o_i, $i = 1\dots 4$, where $(o_i, o_i) \in O$, one can also check that σ is precisely the inverse of h_*. Now let $\mathcal{M}$ be the universal bundle of $G(3,5)$ restricted to S and let

$$\mathbb{P} := \mathbf{P}\mathcal{M}.$$

Then we have the commutative diagram

$$\begin{tikzcd}
 & & \widetilde{S \times \mathbf{P}^2} \arrow[dl, "\alpha"'] \arrow[dr, "\epsilon_2"] & \\
\mathbf{P}^2 \times \mathbf{P}^2 \arrow[r, dashed, "h_* \times id"] & S \times \mathbf{P}^2 \arrow[rr, dotted, "\epsilon"] \arrow[dr, dotted, "h'"'] & & \mathbb{P} \arrow[dl, "h_{\mathbb{P}}"] \\
 & & \mathbf{P}^4 &
\end{tikzcd}$$

where $h = h' \circ (h_* \times id)$, $h_{\mathbb{P}}$ is the tautological map of $\mathbb{P}$ and ϵ is birational. $h_{\mathbb{P}} : \mathbb{P} \to \mathbf{P}^4$ is a morphism of degree two, branched on a very interesting singular quartic threefold considered in [SR] and $\mathfrak{s}_5$-invariant.

Let $E_i = \sigma^{-1}(o_i)$, α is the blow up of $\cup_{i=1\dots 4} E_i \times \{o_i\}$ and ϵ_2 is a divisorial contraction. A main point is that the strict transform by $\epsilon \circ (h_* \times d)$ of $|\mathcal{I}_O(1,1)|$ is the tautological linear system $|\mathcal{O}_{\mathbb{P}}(1)|$ [FV4]. This implies that

Proposition 3.13 *$|\mathcal{O}_{\mathbb{P}}(2)|$ is the strict transform by the map $\epsilon \circ (h_* \times d)$ of the 15-dimensional linear system of nodal conic bundles $|\mathcal{I}^2_O(2,2)|$.*

Let $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ be general, then its discriminant C_Q is a general element of $|-2K_S|$. It is endowed with the étale 2:1 cover $\pi_Q : \tilde{C}_Q \to C_Q$ due to the conic bundle structure. Thus a commutative diagram follows:

$$\begin{tikzcd}
|\mathcal{O}_{\mathbb{P}}(2)| \arrow[r, "d"] \arrow[d, "r"'] & |-2K_S| \arrow[d, "m"'] \\
\mathcal{R}_6 \arrow[r, "f"] & \mathcal{M}_6
\end{tikzcd}$$

where d is the discriminant map $Q \to C_Q$, f is the forgetful map, m is the moduli map and r is the map sending Q to the moduli point of π_Q. This is of independent interest and, hopefully, it will be reconsidered elsewhere.

4 $\mathcal{A}_p$ and the Prym Map

4.1 Prym Varieties and the Prym Map

Let (C, η) be a smooth Prym curve and $\pi : \tilde{C} \to C$ be the étale double cover defined by η, then $\tilde{C}$ has genus $2g-1$ but depends on $3g-3$ moduli. Hence $\tilde{C}$ is not general and, in particular, Brill-Noether theory does not apply a priori to it. In this section we recall the basic results of this theory, modified for the case of $\tilde{C}$, and define the Prym variety of (C, η). Let

$$Nm : Pic^d\,\tilde{C} \to Pic^d\,C,$$

be the *Norm map*, which is defined as follows $Nm(\mathcal{O}_{\tilde{C}}(a)) := \mathcal{O}_C(\pi_* a)$. Nm is clearly surjective and, for $d = 0$, is a morphism of abelian varieties

$$Nm : Pic^0\tilde{C} \to Pic^0 C.$$

Then dim $Ker\ Nm = g - 1$ and its connected component of zero

$$P(C, \eta) := (KerNm)^0$$

is a $g-1$-dimensional abelian variety. $P(C, \eta)$ is known as *the Prym variety of* (C, η). Moreover it is endowed with a natural principal polarization.

Following Mumford's foundation of the theory of Prym varieties we want now to see a different, and very convenient, construction for this abelian variety and its principal polarization.

How many connected components do we have for *Ker Nm*? The answer follows from the exact sequence of 2-torsion groups

$$0 \to < \eta > \to Pic^0_2\,C \xrightarrow{\pi^*} Pic^0_2\,\tilde{C} \xrightarrow{Nm} Pic^0_2\,C \to 0.$$

This implies that $(Ker\ Nm)_2$ has order 2^{2g-1}. Since $\dim P(C, \eta) = g - 1$ then $P(C, \eta)_2$ has order 2^{2g-2} and hence index 2 in $KerNm$. Therefore the connected components are two and every fibre of $Nm : Pic^d\,\tilde{C} \to Pic^d\,C$ is the disjoint union of two copies of $P(C, \eta)$.

Following [M8] it is convenient to fix $d = 2g - 2$ and to study the map

$$Nm : Pic^{2g-2}\tilde{C} \to Pic^{2g-2}C$$

and, in particular, its fibre over the canonical class $o \in Pic^{2g-2}C$. This is

$$Nm^{-1}(o) = \{\tilde{L} \in Pic^{2g-2}\tilde{C} \mid Nm\,\tilde{L} \cong \omega_C\}.$$

In this case the splitting of $Nm^{-1}(o)$ in two copies of $P(C,\eta)$ is ruled by the parity of $h^0(\tilde{L})$. Denoting these copies by P and P^- we have:

- $P := \{\tilde{L} \in Nm^{-1}(o) \;/\; h^0(\tilde{L})$ *is even*$\}$,
- $P^- := \{\tilde{L} \in Nm^{-1}(o) \;/\; h^0(\tilde{L})$ *is odd*$\}$.

Now let $\tilde{g} = 2g - 1$ be the genus of $\tilde{C}$. Since $\tilde{g} - 1 = 2g - 2$ a natural copy of the theta divisor of $Pic^{2g-2}\tilde{C}$ is provided by the Brill Noether-locus

$$W^0_{2g-2}(\tilde{C}) := \tilde{\Theta}.$$

By Riemann singularity theorem and the definition of P, the intersection $P \cap W^0_{2g-2}$ is entirely contained in the singular locus $W^1_{2g-2}(\tilde{C})$ of $\tilde{\Theta}$. Building on this remark it turns out that the scheme

$$\Xi := P \cdot W^1_{2g-2}(\tilde{C})$$

is a principal polarization on P [M8, ACGH]. Notice also that $P \cdot \tilde{\Theta} = 2\Xi$. From a smooth Prym curve (C,η) we can therefore define a principally polarized abelian variety (P,Ξ) of dimension $g-1$.

Definition 4.1 The pair (P,Ξ) is the Prym variety of (C,η).

With a slight abuse $P(C,\eta)$ we will also denote (P,Ξ). The loci we have considered so far in $Nm^{-1}(o)$ are examples of *Prym Brill-Noether loci*.

Definition 4.2 The r-th Prym Brill-Noether scheme $P^r(C,\eta)$ is

- $P \cdot W^r_{2g-2}(\tilde{C})$ if $r+1$ is even,
- $P^- \cdot W^r_{2g-2}(\tilde{C})$ if $r+1$ is odd.

The following properties, in analogy to Brill-Noether theory, are satisfied by the Prym Brill-Noether loci $P^r(C,\eta) \subset P \cup P^-$, see [We]

Theorem 4.3

(1) Let (C,η) be any smooth Prym curve: if $\binom{r+1}{2} \leq g-1$ then $P^r(C,\eta)$ is not empty of codimension $\leq \binom{r+1}{2}$.

(2) Let (C,η) be a general smooth Prym curve and let $\binom{r+1}{2} \leq g-1$ then

- *$P^r(C,\eta)$ has codimension $\binom{r+1}{2}$,*
- *it is irreducible if* $\dim P(C,\eta) > 0$*,*
- *Sing $P^r(C,\eta) = P^{r+2}(C,\eta)$.*

After the family of Jacobians of curves of genus g, the family of Prym varieties of dimension $g-1$ is of special interest in the family of all principally polarized abelian varieties of the same dimension. The reason is that Prym varieties, due to the way they are constructed, can be investigated by means of the theory of curves.

For instance the theta divisor Ξ of a general Prym $P(C,\eta)$ is singular if $\dim P(C,\eta) \geq 6$. Its singular locus is precisely the Prym Brill-Noether locus $P^3(C,\eta) \subset \Xi$. By the way a general point of it is a quadratic singularity with quadratic tangent cone of rank 6.

Prym varieties are specially important in low dimension p. Let $\mathcal{A}_p$ be the moduli space of principally polarized abelian varieties of dimension p. Indeed most knowledge on $\mathcal{A}_p$ for $p \leq 5$ is due to the next property:

Theorem 4.4 *A general ppav of dimension ≤ 5 is a Prym variety.*

This theorem follows considering the Prym map:

Definition 4.5 The Prym map $P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$ is the map sending the moduli point of (C,η) to the moduli point of its associated Prym $P(C,\eta)$.

One can actually show that P_g is dominant for $g \leq 6$. This follows by showing that the tangent map of P_g is generically surjective for $g \leq 6$. To sketch the proof of this latter fact let us briefly recall the following.

For $i^* : H^0(\omega_{\tilde{C}}) \to H^0(\omega_{\tilde{C}})$ the eigenspaces decomposition is

$$H^0(\omega_{\tilde{C}}) = H^+ \oplus H^-,$$

with $H^+ = \pi^* H^0(\omega_C)$ and $H^- = \pi^* H^0(\omega_C \otimes \eta)$.

As is well known the Prym variety P is the image of the map

$$1 - i^* : Pic^0 \tilde{C} \to Pic^0\ \tilde{C}$$

so that its cotangent bundle is canonically isomorphic to $\mathcal{O}_P \otimes H^-$.

For the cotangent space to $\mathcal{R}_g$ at $x := [C,\eta]$ we have the canonical identifications $T^*_{\mathcal{R}_g,x} = T^*_{\mathcal{M}_g,[C]} = H^0(\omega_C^{\otimes 2})$. On the other hand we have $T^*_{\mathcal{A}_{g-1},y} = Sym^2 H^0(\omega_C \otimes \eta)$ for the cotangent space to $\mathcal{A}_{g-1}$ at $y = [P(C,\eta)]$.

Finally it turns out that the multiplication map

$$\mu : Sym^2 H^0(\omega_C \otimes \eta) \to H^0(\omega_C^{\otimes 2})$$

is the cotangent map of P_g at x, see for instance [Be3]. Therefore P_g is dominant at x iff μ is injective. It is well known that, for a general Prym curve (C,η) of genus g, μ has maximal rank, that is either it is injective or surjective. Counting dimensions we conclude that

Theorem 4.6 *μ is injective for a general (C,η) iff $g \leq 6$.*

For $g \geq 7$ the map μ is surjective for a general (C,η), that is, the tangent map of P_g is injective at a general point. The Prym-Torelli theorem says that P_g is generically injective for $g \leq 7$. The description of the loci where P_g fails to be injective is in many respects an open problem.

Revisiting the fibres of the Prym map P_g for $g \leq 6$ provides a remarkable sequence of beautiful geometric constructions. For brevity we only mention, for $3 \leq g \leq 6$, a geometric description of a general fibre. Also, we omit to discuss the extension of P_g to $\overline{\mathcal{R}}_g$. The fibre over $[P, \Xi] \in \mathcal{A}_{g-1}$ is birationally described as follows, see [Ve2, Re, Do2]

- $g = 3$. *The Siegel modular quartic threefold* $|2\Xi|/\mathbb{Z}_2^4$.
- $g = 4$. *The Kummer variety* $P/ < -1 >$.
- $g = 5$. *The double cover of the Fano surface of a cubic threefold.*

By a famous result of Donagi and Smith, the Prym map in genus 6 has degree 27 and its monodromy group is the Weyl group of the lattice E_6. The configuration of a general fibre of

$$p_6 : \mathcal{R}_6 \to \mathcal{A}_5$$

is the one of 27 lines on a smooth cubic surface [DS, Do2]. The ramification divisor $\mathcal{D}$ is the locus of elements $[C, \eta] \in \mathcal{R}_6$ such that μ is not an isomorphism. Equivalently the image of C in $\mathbf{P}^4$ under the map defined by $|\omega_C \otimes \eta|$ is contained in a quadric. A general fibre of p_6 over the branch divisor $p_6(\mathcal{D})$ has the configuration of the lines of a cubic surface with an ordinary double point: 6 points of simple ramification and 15 unramified points.

4.2 Unirationality of universal Pryms

Since $\mathcal{R}_g$ is unirational and the Prym map is dominant for $g \leq 6$, we have

Theorem 4.7 *$\mathcal{A}_p$ is unirational for $p \leq 5$.*

Let us review more in detail what is known for the moduli space $\mathcal{A}_p$:

- *$\mathcal{A}_p$ is rational for $p \leq 3$ (since it is birational to $\mathcal{M}_p$),*
- *$\mathcal{A}_4$ and $\mathcal{A}_5$ are unirational (via the Prym map),*
- *$\mathcal{A}_6$ is a remarkable open problem,*
- *$\mathcal{A}_p$ is of general type for $p \geq 7$ (Mumford, Tai).*

We can also study the universal principally polarized abelian. See [M7, T] for p bigger or equal than 7.

$$v_g : \mathcal{X}_p \to \mathcal{A}_p$$

and its pull-back by the Prym map. We define it via the fibre product

$$\begin{array}{ccc}
\mathcal{P}_{g-1} & \xrightarrow{\chi_{g-1}} & \mathcal{X}_{g-1} \\
{\scriptstyle u_g}\downarrow & & {\scriptstyle v_g}\downarrow \\
\mathcal{R}_g & \xrightarrow{P_g} & \mathcal{A}_{g-1}
\end{array}$$

Definition 4.8 $\mathcal{P}_{g-1}$ is the universal Prym over $\mathcal{R}_g$.

Let $Pic^{inv}_{0,2g-1}$ be the universal Picard variety over the moduli of curves of genus $2g-1$ with a fixed point free involution. To give an equivalent definition let us consider the universal Norm map

$$Nm : Pic^{inv}_{0,2g-1} \to Pic^0_{0,g}.$$

Since the zero section $o : \mathcal{A}_{g-1} \to \mathcal{X}_{g-1}$ is fixed, the universal Prym $\mathcal{P}_{g-1}$ can be also defined as the connected component of o in the Kernel of the universal Norm map. The *unirationality* of $\mathcal{P}_p$ and of $\mathcal{X}_p$ is known for:

- $\mathcal{P}_p$, $p \leq 3$ *(implicit to several constructions)*,
- $\mathcal{P}_4$ *(via Prym Brill-Noether theory* [Ve4]),
- $\mathcal{P}_5$ *(via nodal conic bundles,* [FV4]*)*.

After our previous use of conic bundles over $\mathbf{P}^2$ having a 4-nodal sextic as discriminant curve, we insist in using them to show that

Theorem 4.9 $\mathcal{P}_5$ *is unirational.*

Preliminarily we recall that the n-th *Abel-Prym map* of (C, η) is the map

$$a_n^- : \tilde{C}^n \to Ker\ Nm$$

obtained from the composition of the maps

$$\tilde{C}^n \xrightarrow{a_n} Pic^n \tilde{C} \xrightarrow{1-i^*} KerNm \subset Pic^0 \tilde{C},$$

where a_n is the Abel map, that is, $a_n(x_1, \dots, x_n) = \mathcal{O}_{\tilde{C}}(x_1 + \dots + x_n)$ and $(1 - i^*)(L) = L \otimes i^* L^{-1}$. It is known that the image of a^- is in the connected component of zero $(Ker\ Nm)^0$ iff n is even. Furthermore a_n^- is generically injective for $n \leq g-1$ and dominant for $n \geq g-1$, see [Be1].

Let $\tilde{\mathcal{C}}^n \to \mathcal{R}_g$be the universal product with fibre $\tilde{C}^n$ at $[C, \eta]$ and let

$$\mathfrak{a}_n^- : \tilde{\mathcal{C}}^n \to \mathcal{P}_{g-1}$$

be the universal Abel-Prym map defined as follows:

- n even then $\mathfrak{a}_n^-(x_1, \dots, x_n) = \mathcal{O}_{\tilde{C}}(x_1 - i(x_1) + \dots + x_n - i(x_n))$,
- n odd then $\mathfrak{a}_n^-(x_1, \dots, x_n) = \mathcal{O}_{\tilde{C}}(2x_1 - 2i(x_1) + \dots + x_n - i(x_n))$.

We can now prove that the universal Prym $\mathcal{P}_5$ is unirational. In the spirit of the previous method it will be enough to use the convenient linear system $\mathbf{P}^{15}$ of singular $(2, 2)$ hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^2$ considered in Sect. 3.4. We will show that $\mathbf{P}^{15}$ dominates $\mathcal{P}_5$. Putting $n = 5$ and $g = 6$ it is clear that $\mathfrak{a}_5^- : \tilde{\mathcal{C}}^5 \to \mathcal{P}_5$ is dominant. Therefore we are left to show that

Theorem 4.10 *$\tilde{\mathcal{C}}^5$ is unirational.*

Proof Let $[\tilde{C}; x_1, \ldots, x_5]$ be an element of $\tilde{\mathcal{C}}^5$. We know that $\tilde{C}$ parametrizes the family of lines which are components of the singular conics of a conic bundle $T \in \mathbf{P}^{15}$. Therefore x_i corresponds to a line of this type, say $l_i \subset T$, $i = 1 \ldots 5$. Since three conditions are needed so that some $T \in \mathbf{P}^{15}$ contains l_i, it follows that $\tilde{\mathcal{C}}^5$ is dominated by the family of 5-tuples of lines as above. On the other hand it is clear that the Hilbert scheme of these lines is $\mathbf{P}^2 \times \mathbf{P}^{2*}$. Hence it follows that there exists a dominant rational map

$$(\mathbf{P}^2 \times \mathbf{P}^{2*})^5 \to \tilde{\mathcal{C}}^5.$$

This completes the proof of the unirationality of $\mathcal{P}_5$. □

4.3 Testing the slope of $\overline{\mathcal{A}}_p$ in low Genus

It is time to compactify the Prym map and use the parametrizations of $\mathcal{R}_g$ we have constructed, in order to study the slope of $\overline{\mathcal{A}}_p$ and, possibly, deduce further properties in low genus. In what follows $\overline{\mathcal{A}}_p$ denotes one of the toroidal compactifications in use, namely the first Voronoi or perfect cone compactification. See [AMRT] and [SB1] for a fundamental account.

The study of the slope performed here includes the case of $\overline{\mathcal{A}}_6$. We reach a lower bound of its slope using families of rational curves which are sweeping the boundary divisor. This just means that the union of these curves contains a dense open subset of the boundary divisor. The latter is actually dominated by the universal Prym $\mathcal{P}_5$ and hence by the rational parametrization of $\mathcal{P}_5$ we have constructed. Let us consider the extended Prym map

$$\overline{P}_g : \overline{\mathcal{R}}_g \to \overline{\mathcal{A}}_{g-1}$$

induced by P_g. We have already considered the moduli of quasi stable Prym curves $\overline{\mathcal{R}}_g$. On the other hand it turns out that $\overline{\mathcal{A}}_p$ is the blowing up of the Satake compactification

$$\mathcal{A}_p^s = \mathcal{A}_p \sqcup \mathcal{A}_{p-1} \sqcup \ldots \sqcup \mathcal{A}_0$$

along its boundary, cfr. [SB1]. It is well known that the exceptional divisor

$$D_p \subset \overline{\mathcal{A}}_p$$

of such a blowing up is integral, moreover

$$CH^1(\overline{\mathcal{A}}_p) = \mathbb{Z}\lambda_p \oplus \mathbb{Z}\delta_p$$

where λ_p is the Hodge class and δ_p is the class of the boundary D_p. Let $E \subset \overline{\mathcal{A}}_p$ be an effective divisor of class $a\lambda_p - b\delta_p$, such that $a, b > 0$. We define its slope as $s(E) := \frac{a}{b}$.

Definition 4.11 The slope of $\overline{\mathcal{A}}_p$ is $s(\overline{\mathcal{A}}_p) := min\{s(E)\ /\ E \text{ as above}\}$.

We recall that $s(\overline{\mathcal{A}})_p$ governs the pseudoeffectiveness of the canonical class of $\overline{\mathcal{A}}_p$ and that, as a consequence of the results in [BDPP], it follows:

- $s(\overline{\mathcal{A}}_p) > p + 1$ *implies that* $\overline{\mathcal{A}}_p$ *is uniruled*,
- $s(\overline{\mathcal{A}}_p) < p + 1$ *implies that* $\overline{\mathcal{A}}_p$ *is of general type*.

For $p \leq 6$ this is the situation, to be partially presented here:

- $s(\overline{\mathcal{A}}_4) = 8$ [GSM]
- $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$ [FGSMV]
- $s(\overline{\mathcal{A}}_6) \geq 5, 3$ [FV4]

To sketch a description of D_p let us fix a ppav P of dimension $p - 1$. As is well known the family of algebraic groups defined by the extensions

$$0 \to \mathbf{C}^* \to A \to P \to 0$$

defines a map $j_P : P \to D_p$ whose image is birational to the Kummer variety $P/ < -1 >$, cfr. [AMRT]. Since P is a fibre of $u_{p-1} : \mathcal{X}_{p-1} \to \mathcal{A}_{p-1}$, the constructions yields a dominant map

$$j : \mathcal{X}_{p-1} \to D_p.$$

It is therefore natural to test the slope of $\overline{\mathcal{A}}_p$ using families of curves R sweeping $\overline{\mathcal{X}}_{p-1}$. From curves like R one can try to compute a lower bound of $s(\overline{\mathcal{A}}_p)$, using the inequality $R \cdot j^*E \geq 0$ for E effective of class $a\lambda_p - b\delta_p$ with $a, b > 0$. Let ρ be the class of R. Of course this implies:

$$s(\overline{\mathcal{A}}_p) \geq \frac{\deg \rho \cdot j^*\delta_p}{\deg \rho \cdot j^*\lambda_p}.$$

In order to describe some intersection classes for $p = 6$, we conclude this section with a few technical remarks. Let $\tilde{\mathcal{R}}_g$ be the complement in $\overline{\mathcal{R}}_g$ to $\Delta_i \cup \Delta_{i:g-i}$, $i \geq 1$. It will be sufficient to work in $\tilde{\mathcal{R}}_g$. Let $\tilde{\mathcal{P}}_g$ be the pull-back of $\tilde{\mathcal{R}}_g$ by $u_g : \mathcal{P}_g \to \overline{\mathcal{R}}_g$ and let $\tilde{\mathcal{A}}_g := \overline{\mathcal{A}}_g - D_g$. Then the diagram

$$\begin{array}{ccccc} \tilde{\mathcal{P}}_{g-1} & \xrightarrow{\chi} & \tilde{\mathcal{X}}_{g-1} & \xrightarrow{j} & \overline{\mathcal{A}}_g \\ {\scriptstyle u_g}\downarrow & & {\scriptstyle v_g}\downarrow & & \\ \tilde{\mathcal{R}}_g & \xrightarrow{P_g} & \tilde{\mathcal{A}}_{g-1} & & \end{array}$$

is commutative. The next formulae are known. Let $\theta \in CH^1(\widetilde{\mathcal{X}}_{g-1})$ be the class of the universal theta divisor, trivialized along the zero section. Let $\theta_{pr} := \chi^*(\theta) \in CH^1(\tilde{\mathcal{P}}_g)$, then the next formulae hold for g even:

- $j^*([D_g]) = -2\theta + v^*_{g-1}([D_{g-1}]) \in CH^1(\widetilde{\mathcal{X}}_{g-1})$.
- $(j \circ \chi)^*(\lambda_g) = u^*_g\big(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\big) \in CH^1(\tilde{\mathcal{P}}_{g-1})$.
- $(j \circ \chi)^*([D_g]) = -2\theta_{pr} + u^*_g(\delta'_0) \in CH^1(\tilde{\mathcal{P}}_{g-1})$,

see [GZ, FV4]. We can also put into play the Abel-Prym map

$$\mathfrak{a}^-_{g-1} : \tilde{\mathcal{C}}^{g-1} \to \tilde{\mathcal{P}}_{g-1}.$$

In $CH^1(\tilde{\mathcal{C}}^n)$ we have the ψ-classes $\psi_{x_1} \dots \psi_{x_n}$, defined by the cotangent spaces at $x_1, \dots, x_n$ in the pointed curve $(\tilde{C}; x_1, \dots, x_n)$. One can compute the class of $(\mathfrak{a}^-_{g-1})^*\theta_{pr}$ in $CH^1(\tilde{\mathcal{C}}^{g-1})$ as in [FV4]:

$$\mathfrak{a}^-_{g-1}{}^*\theta_{pr} = \frac{1}{2}\sum_{j=1}^{g-2} \psi_{x_j} + 2\psi_{x_{g-1}} + 0 \cdot \Big(\lambda + (\mathfrak{a}^-_{\mathfrak{g}-1} \circ u_g)^*(\delta'_0 + \delta''_0 + \delta_0^{\mathrm{ram}})\Big) + \dots$$

We want only to point out that $\lambda, \delta'_0, \delta''_0, \delta_0^{\mathrm{ram}}$ have zero coefficient.

4.4 On the slope and boundary of $\overline{\mathcal{A}}_6$

The previous analysis of divisorial classes can be in principle used to study the slope of $\overline{\mathcal{A}}_p$ or other properties. This is our program for $p = 6$. A geometric basis for it also exists. This is provided by the unirationality of $\mathcal{P}_5$ and by the linear system of conic bundles

$$\mathbf{P}^{15} = |\mathcal{O}_{\mathbb{P}}(2)| = |\mathcal{I}^2_O(2,2)|$$

that we already considered in Sect. 3. We keep the previous notation: in particular $\mathbb{P}$ is the $\mathbf{P}^2$-bundle we know over the quintic Del Pezzo surface S. Let $\mathbb{L}$ be the Hilbert scheme of lines which are in the fibres of $\mathbb{P} \to S$. Then $\mathbb{L}$ is just $\mathbb{P}^*$ and we can define a dominant rational map as follows

$$q : \mathbb{L}^5 \to \tilde{\mathcal{C}}^5.$$

For a general $x := (l_1, \dots, l_5) \in \mathbb{L}^5$ a unique $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ contains $l_1 \dots l_5$. Hence x defines a point $y \in \tilde{C}^5$, where $\tilde{C}$ is the family of the lines in the singular fibres of Q. By definition $q(x) := y$. Therefore we have a sequence of dominant rational maps

$$\mathbb{L}^5 \xrightarrow{q} \tilde{\mathcal{C}}^5 \xrightarrow{\mathfrak{a}^-_5} \mathcal{P}_5 \xrightarrow{j \circ P_5} \mathcal{D}_6 \subset \overline{\mathcal{A}}_6.$$

Let $\phi := j \circ \mathfrak{a}_5^- \circ q$, then ϕ is dominant. Now we choose a family of maps

$$f : \mathbf{P}^1 \to D_6$$

such that the curve $f_*(\mathbf{P}^1)$ moves in a family sweeping D_6. Previously, it will be useful to know some numerical characters of the linear system $|\mathcal{O}_{\mathbb{P}}(2)|$, see [FV4]:

Lemma 4.12 *For a smooth $Q \in \left|\mathcal{O}_{\mathbf{P}}(2)\right|$, we have that $\chi_{top}(Q) = 4$, whereas $\chi_{top}(Q) = 5$ if Sing Q is an ordinary double point.*

Lemma 4.13 *In a Lefschetz pencil $P \subset |\mathcal{O}_{\mathbf{P}}(2)|$ there are precisely 77 singular conic bundles and 32 conic bundles with a double line.*

A family of maps f is constructed as follows: fix a general configuration (l_1, l_2, l_3, l_4, o) of four lines $l_1 \dots l_4$ in the fibres of $\mathbb{P}$ and a point $o \in \mathbb{P}$. Let $\mathbf{P}^1$ be the pencil of lines through o in the fibre of $\mathbb{P}$ containing o. Each line $l \in \mathbf{P}^1$ defines the element $\phi(l_1, l_2, l_3, l_4, l) \in D_6$. This defines a map

$$f : \mathbf{P}^1 \to D_6 \subset \overline{\mathcal{A}}_6$$

and the family of curves $f(\mathbf{P}^1)$ is sweeping D_6. Such a family and the characters of a Lefschetz pencil P of conic bundles are the geometric support for the intersection classes count we are going to outline. For the complete set of these computations see [FV4, Sects. 3 and 4] .

Let $m : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$ be the moduli map sending (l_1, l_2, l_3, l_4, l) to the Prym curve (C, η) which is the discriminant of the conic bundle Q. Moreover consider also the natural map $q : \mathbf{P}^1 \to \tilde{\mathcal{C}}^5$. After more work we obtain:

$$m^*\lambda = 9 \times 6\ ,\ m^*\delta_0' = 3 \times 77\ ,\ m^*\delta_0^{\mathrm{ram}} = 3 \times 32\ ,\ m^*\delta_0'' = 0\ ,\ q^*\psi_{l_j} = 9.$$[4]

We use these data to bound the slope of $\overline{\mathcal{A}}_6$: at first consider the effective class $\rho := [f_*(\mathbf{P}^1)] \in NE_1(\overline{\mathcal{A}}_6)$, one can compute that

$$\rho \cdot \lambda_6 = q_*(\mathbf{P}^1) \cdot (\mathfrak{a}_{\mathfrak{g}-1}^- \circ u_g)^* \Big(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\Big) = 6 \times 9 - \frac{3 \times 32}{4} = 30, \quad \text{and}$$

$$\begin{aligned} \rho \cdot [D_6] &= -q_*(\mathbf{P}^1) \cdot \Big(\sum_{j=1}^{4} \psi_{x_j} + 4\psi_{x_5}\Big) + i_*(\mathbf{P}^1) \cdot (\mathfrak{a}_{\mathfrak{g}-1}^- \circ u_g)^*(\delta_0') \\ &= -8 \times 9 + 3 \times 77 = 159. \end{aligned}$$

[4]To simplify the notation we identify $Pic(\mathbf{P}^1)$ to $\mathbb{Z}$ via the degree map.

Since ρ is the class of a family of curves sweeping D_6, it follows that $\rho \cdot E \geq 0$ for every effective divisor E. This implies the bound

$$s(\overline{\mathcal{A}}_6) \geq \frac{\rho \cdot [D_6]}{\rho \cdot \lambda_6} = 5, 3.$$

5 Prym Curves and K3 Surfaces

5.1 Mukai Constructions and universal Jacobians

In order to discuss the uniruledness of some moduli spaces as $\mathcal{M}_g$, $\mathcal{R}_g$ or $\mathcal{A}_g$, we have made a large use of constructions involving rational families of curves on rational surfaces. On the other hand we have often remarked that further constructions definitely come into play in the historical and scientific evolution of this subject. Now the next constructions to be considered are Mukai constructions for K3 surfaces and canonical curves in low genus.

As it is well known, these can be very well used to deduce the unirationality of $\mathcal{M}_g$ for $7 \leq g \leq 9$ and $g = 11$. Furthermore they represent a very natural motivation for this result. To add a dubious speculation, it is possibly not excluded that the rationality problem for $\mathcal{M}_g$ could be approached via these constructions and Geometric Invariant Theory, in the range $7 \leq g \leq 9$. For $g \leq 9$, the unirationality of the universal Picard variety $Pic_{d,g}$ also follows from these constructions, cfr. [Mu1, Mu2, Ve1]

However we concentrate in this part on some related but different constructions which are natural and useful to study the Prym moduli space $\mathcal{R}_g$ in low genus. For every g we consider a special family of K3 surfaces of genus g, namely the family of *Nikulin surfaces*.

To improve our study of $\mathcal{R}_g$ we rely on these surfaces, their hyperplane sections and their moduli. The geometric description of Nikulin surfaces in low genus g presents some unexpected and surprising analogies to Mukai constructions, cfr. [FV3, FV6, Ve5].

Definition 5.1 A K3 surface of genus g is a pair (S, L) such that S is a K3 surface and $L \in Pic\ S$ is primitive, big and nef, and $c_1(L)^2 = 2g - 2$.

The *moduli space of* (S, L) is denoted by $\mathcal{F}_g$, it is quasi projective and integral of dimension 19. Let $g \geq 3$ and (S, L) general. Then L is very ample and defines an embedding $S \subset \mathbf{P}^g$ as a surface of degree $2g - 2$, cfr. [Huy]. The next result is due to Mukai and Mori [MM]:

Theorem 5.2 *A general canonical curve C of genus g is a hyperplane section of a K3 surface $S \subset \mathbf{P}^g$ iff $g \leq 11$ and $g \neq 10$.*

Since the moduli map $m : |\mathcal{O}_S(1)| \to \mathcal{M}_g$ turns out to be not constant, the theorem implies the *uniruledness* of $\mathcal{M}_g$ for $g \leq 11$, $g \neq 10$. Actually the theorem fits in the nice series of geometric constructions discovered by Mukai. They relate K3 surfaces, canonical curves in low genus and homogenous spaces. Here is the list of homogeneous spaces in use. A space in the list is denoted by $\mathbb{S}_g$ and it is embedded in $\mathbf{P}^{N_g}$ by the ample generator $\mathcal{O}_{\mathbb{S}_g}(1)$ of $Pic\ \mathbb{S}_g$:

- $\mathbb{S}_6$ *is the Grassmannian* $G(2,5)$ *embedded in* $\mathbf{P}^9$,
- $\mathbb{S}_7$ *is the orthogonal Grassmannian* $OG(5,10)$ *embedded in* $\mathbf{P}^{15}$,
- $\mathbb{S}_8$ *is the Grassmannian* $G(2,6)$ *embedded in* $\mathbf{P}^{14}$,
- $\mathbb{S}_9$ *is the symplectic Grassmannian* $SO(3,6)$ *embedded in* $\mathbf{P}^{13}$,
- $\mathbb{S}_{10}$ *is the* $\mathbb{G}_2$*-homogenous space* G_2 *embedded in* $\mathbf{P}^{13}$.

Mukai constructions are developed in [Mu1, Mu2, Mu3, Mu4]. In particular they imply the next two theorems:

Theorem 5.3 *Let* (S, L) *be a general K3 surface of genus* $g \in [7, 10]$*, then* S *is biregular to a 2-dimensional linear section of* $\mathbb{S}_g \subset \mathbf{P}^{N_g}$ *and* $L \cong \mathcal{O}_S(1)$.

This is often called *Mukai linear section theorem*. We also include:

Theorem 5.4 *Let* $g = 6$*, then a general* S *is a 2-dimensional linear section of a quadratic complex of the Grassmannian* $\mathbb{S}_6 \subset \mathbf{P}^9$ *and* $L \cong \mathcal{O}_S(1)$.

What is the link between Mukai constructions and the unirationality of $\mathcal{M}_g$ in low genus? A direct consequence of these constructions is that

Theorem 5.5 *The universal Picard variety* $Pic_{d,g}$ *is unirational for* $g \leq 9$.

Proof See [Ve1] for details, here we will give a sketch the proof in view of some applications. Let $U \subset \mathbb{S}_g^g$ be the open set of g-tuples of points $x := (x_1, \dots, x_g)$ spanning a space $< x >$ transversal to $\mathbb{S}_g$. Then $C_x := \mathbb{S}_g \cap < x >$ is a smooth, g-pointed canonical curve. Given the non zero integers $d_1, \dots, d_g$ such that $d_1 + \dots + d_n := d$, let $a : C_x^g \to Pic^d\ C$ be the map sending $(y_1, \dots, y_g)$ to $\mathcal{O}_{C_x}(\sum d_i y_i)$. It is well known that a is surjective. Now consider the map $A_g : U \to Pic_{d,g}$ defined as follows: $A_g(x) := [C_x, L_x] \in Pic_{d,g}$, where $L_x := \mathcal{O}_{C_x}(\sum d_i x_i)$. By Mukai results and the latter remark A_g is dominant for $g \leq 9$. Since U is rational it follows that $Pic_{d,g}$ is unirational. □

For $g \geq 10$ the transition of $Pic_{d,g}$ from negative Kodaira dimension to general type has remarkable and nice aspects:

- $Pic_{d,10}$ *has Kodaira dimension* 0,
- $Pic_{d,11}$ *has Kodaira dimension* 19,
- $Pic_{d,g}$ *is of general type for* $g \geq 12$.

This picture summarizes several results, see [BFV, FV1, FV3]. The presence of the number $19 = dim\ \mathcal{F}_{11}$ is not a coincidence. It reflects the Mukai construction for $g = 11$, which implies that a birational model of $\mathcal{M}_{11}$ is a $\mathbf{P}^{11}$-bundle over $\mathcal{F}_{11}$. Building in an appropriate way on this information, it follows that $kod(Pic_{d,11}) = 19$.

It is now useful to summarize some peculiar aspects of the family of hyperplane sections of K3 surfaces of genus g, made visible after Mukai. Let $C \subset S$ be a smooth, integral curve of genus g in a *general* $K3$ surface:

- Parameter count: *C cannot be general for* $g \geq 12$. *One expects the opposite for* $g \leq 11$.
- Genus 10 is unexpected: *if C is a curvilinear section of* $\mathbb{S}_{10}$, *then C is not general and the parameter count is misleading.*
- Genus $g \in [6, 9]$ and $g = 11$ is as expected: *C is a curvilinear section of* $\mathbb{S}_g$ *and general in moduli.*
- Genus 11: *C embeds in a unique K3. Birationally* $\mathcal{M}_{11}$ *is a* $\mathbf{P}^{11}$*-bundle on* $\mathcal{F}_{11}$.
- Genus 10: *Special syzygies: the Koszul cohomology group* $K_2(C, \omega_C)$ *is non zero.*

With respect to the above properties some unexpected analogies appear, as we will see, when considering the family of hyperplane sections of the so called Nikulin surfaces. Nikulin surfaces are K3 surfaces of special type to be reconsidered in the next sections.

5.2 Paracanonical Curves on K3 Surfaces

A smooth integral curve C of genus g is said to be *paracanonical* if it is embedded in $\mathbf{P}^{g-2}$ so that $\mathcal{O}_C(1) \in Pic^{2g-2}(C)$. Let $g \geq 5$, we are now interested to study the following situation

$$C \subset S \subset \mathbf{P}^{g-2},$$

where S is a smooth K3 surface of genus $g-2$ and C is paracanonical. Then $h := c_1(\mathcal{O}_S(1))$ is a very ample polarization of genus $g-2$ and $|C|$ is a linear system of paracanonical curves on S. In particular each $D \in |C|$ is endowed with a degree zero line bundle

$$\alpha_D := \omega_D(-h) \in Pic^0(D).$$

The Picard number $\rho(S)$ of S is at least two. At first we deal with the case $\rho(S) = 2$. Then we specialize to $\rho(S) \geq 2$ in order to construct Nikulin surfaces and see the predicted analogies with Mukai constructions. Let

$$c := c_1(\mathcal{O}_S(C)) \ , \ n := c_1(\mathcal{O}_S(C - H))$$

where $H \in |\mathcal{O}_S(1)|$, then we have

$$\begin{pmatrix} c^2 & 0 \\ 0 & n^2 \end{pmatrix} = \begin{pmatrix} 2g-2 & 0 \\ 0 & -4 \end{pmatrix}$$

The matrix defines an abstract rank 2 lattice embedded in $Pic\ S$. Notice that h, c generate the same lattice and that $hc = c^2 = 2g - 2$ and $h^2 = 2g - 6$. Consistently with definition 5.1, it is perhaps useful to adopt the following

Definition 5.6 A K3 surface (S, L) of genus g is d-nodal if $Pic\ S$ contains a primitive vector n orthogonal to $c_1(L)$ and such that $n^2 = -2d \leq -2$.

If $d = 1$ then n or $-n$ is an effective class, for good reasons this is said to be a nodal class. For $d \geq 2$ the classes n and $-n$ are in general not effective. The locus, in the moduli space $\mathcal{F}_g$, of d-nodal K3 surfaces is an integral divisor whose general element is a K3 surface of genus g and Picard lattice as above, cfr. [Huy]. It will be denoted as

$$\mathcal{F}_{g,d}.$$

We are only interested to the case $n^2 = -4$, so we assume the latter equality from now on. Let (S, L) be general 2-nodal and $c = c_1(L)$, then

$$Pic(S) = \mathbb{Z}c \oplus \mathbb{Z}n.$$

We notice from [BaV] the following properties: $h^i(\mathcal{O}_S(n)) = 0$, $i = 0, 1, 2$, and $h^0(\mathcal{O}_D(n)) = 0$, $\forall D \in |C|$. Moreover every $D \in |C|$ is an integral curve. Then we consider the compactified universal Picard variety

$$j : \mathcal{J} \to |C|,$$

whose fibre at a general element $D \in |C|$ is $Pic^0(D)$. Interestingly the map sending D to $\alpha_D := \omega_D(-h) \in Pic^0(D)$ defines a regular section

$$s : |C| \to \mathcal{J}.$$

Now we have in $\mathcal{J}$ the locus

$$\mathcal{J}_m := \{(D, A) \in \mathcal{J} \ / \ h^0(A^{\otimes m}) \geq 1\}$$

If D is smooth, the restriction of $\mathcal{J}_m$ to the fibre $Pic^0(D)$ of j is precisely the m-torsion subgroup $Pic^0_m(D)$. What is the scheme $s^*(\mathcal{J}_m)$?

Theorem 5.7

- *Let s be transversal to $\mathcal{J}_m$, then $s^*\mathcal{J}_m$ is smooth of length $\binom{2m^2-2}{g}$.*
- *Assume (S, L) is general then s is transversal to $\mathcal{J}_m$ and all the elements $D \in s^*\mathcal{J}_m$ are smooth curves.*

Proof We sketch the proof of the first statement, see [BaV]. The nice proof of the second statement is much more recent, see [FK]. Let $D \in |C|$ and let $N \in Div(S)$ be a divisor of class n. Consider the standard exact sequence

$$0 \to \mathcal{O}_S(mN - C) \to \mathcal{O}_S(mN) \to \mathcal{O}_D(mN) \to 0$$

and its associated long exact sequence. Then consider the cup product

$$\mu : H^1(\mathcal{O}_S(mN - C)) \otimes H^0(\mathcal{O}_S(C)) \to H^1(\mathcal{O}_S(mN)).$$

Let us set $\mathbf{P}^a := \mathbf{P}H^1(\mathcal{O}_S(mN - C))$, $\mathbf{P}^b := \mathbf{P}H^0(\mathcal{O}_S(C))$ and $\mathbf{P}^{ab+a+b} := \mathbf{P}(H^1(\mathcal{O}_S(mN - C)) \otimes H^0(\mathcal{O}_S(C)))$. We have the Segre embedding

$$\mathbf{P}^a \times \mathbf{P}^b \subset \mathbf{P}^{ab+a+b}.$$

It turns out that $s^*(\mathcal{J}_m)$ can be viewed as the intersection scheme

$$\mathbf{P}(Ker\ \mu) \cdot (\mathbf{P}^a \times \mathbf{P}^b).$$

One can check that $a = \max\{0, 2m^2 - 2 - g\}$, $b = g$ and $codim\, \mathbf{P}(Ker\ \mu) = a + b$. Hence the length of $s^* \mathcal{J}_m$ is the degree of $\mathbf{P}^a \times \mathbf{P}^b$, that is $\binom{2m^2-2}{g}$. □

Remark 5.8 We point out that the *most non transversal* situation is possible: $|C| = s^* \mathcal{J}_m$. See the next discussion.

5.3 Mukai Constructions and Nikulin Surfaces

We want to discuss the case $m = 2$, that is to study the set $s^{-1}(\mathcal{J}_2)$ and some related topics. Preliminarily we remark that a smooth $C \in s^{-1}(\mathcal{J}_2)$ is embedded with S in $\mathbf{P}^{g-2}$ as a Prym canonical curve of genus g. This just means that $\mathcal{O}_C(1)$ is isomorphic to $\omega_C \otimes \eta$, with $\eta \ncong \mathcal{O}_C$ and $\eta^{\otimes 2} \cong \mathcal{O}_C$.

By the previous formula the scheme $s^* \mathcal{J}_2$ has length $\binom{6}{g}$, provided it is 0-dimensional. This immediately implies that

Lemma 5.9 *The set $s^{-1}(\mathcal{J}_2)$ is either empty or not finite for $g \geq 7$.*

Let $m_2 : Sym^2 H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_C(2H))$ be the multiplication map, where $H \in |\mathcal{O}_S(1)|$. We notice the next lemma, whose proof is standard.

Lemma 5.10 *m_2 is surjective iff $h^1(\mathcal{O}_S(2H - C)) = 0$.*

Notice also that, since C is embedded by a non special line bundle, the surjectivity of m_2 is equivalent to the surjectivity of the multiplication

$$m_k : Sym^k H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_C(kH))$$

for $k \geq 1$ [ACGH] ex. D-5 p. 140. In other words a smooth C is projectively normal iff $h^1(\mathcal{O}_S(2H - C)) = 0$. Consider the standard exact sequence

$$0 \to \mathcal{O}_S(2N - C) \to \mathcal{O}_S(2N) \to \mathcal{O}_C(2N) \to 0$$

where $N = C - H$. Then observe that the divisor $2N - C$ is not effective. Indeed H is very ample and we have $(2N - C)\cdot H = 10 - 2g \leq 0$ under our assumption that $g \geq 5$. From this the non effectiveness of $2N - C$ follows. Moreover we have $H^1(\mathcal{O}_S(2N - C)) \cong H^1(\mathcal{O}_S(2H - C))$ by Serre duality. Hence, passing to the associated long exact sequence, we have

$$0 \to H^0(\mathcal{O}_S(2N)) \to H^0(\mathcal{O}_C(2N)) \to H^1(\mathcal{O}_S(2H - C)).$$

This implies the next statement:

Theorem 5.11 *Assume $h^1(\mathcal{O}_S(2H - C)) = 0$, then it follows that*

$$h^0(\mathcal{O}_S(2N)) = 1 \Longleftrightarrow s^{-1}(\mathcal{J}_2) = |C|$$

We can start our discussion from this statement. It is obvious that the condition $h^0(\mathcal{O}_S(2N)) = 1$ implies $s^{-1}(\mathcal{J}_2) = |C|$, without any assumption. For the purposes of this section, and to avoid technical details, it will be enough to discuss the existence and the features of the families of K3 surfaces as above which satisfy the following condition:

∘ $h^0(\mathcal{O}_S(2N)) = 1$ *and the unique curve $E \in |2N|$ is smooth.*

Then the curve E immediately highlights some beautiful geometry of the surface S. From $E^2 = -16$ and $HE = 8$ one can easily deduce that:

Proposition 5.12 *E is the union of eight disjoint lines $E_1 \dots E_8$.*

Actually these surfaces exist and their families have been studied by many authors, in particular by Nikulin in [N], by van Geemen and Sarti in [SvG] and by Garbagnati and Sarti in [GS]. Using the curve E we can introduce them as follows. Since $E \sim 2N$ we have the 2:1 covering $\pi : \tilde{S}' \to S$ branched on E. From π_S we have the commutative diagram

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{\sigma'} & \tilde{S} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ S & \xrightarrow{\sigma} & \overline{S} \end{array}$$

Here σ is the contraction of the -2-lines $E_1 \dots E_8$ to eight nodes. Moreover σ' is the contraction of the exceptional lines $\pi^{-1}(E_i)$, $i = 1 \dots 8$ and finally $\overline{\pi}$ is the $2:1$ cover branched over the even set of nodes $Sing\ \overline{S}$.

It is easy to see that $\tilde{S}$ is a smooth and minimal K3 surface. Actually $\pi_{\overline{S}}$ is the quotient map of a symplectic involution $i : \tilde{S} \to \tilde{S}$ on $\tilde{S}$. The fixed points of i are eight, by Lefschetz fixed point theorem. See [N, SvG, GS].

Definition 5.13 Let S be a K3 surface and $n \in Pic\ S$. n is a Nikulin class if $2n \sim E_1 + \cdots + E_8$, where $E_1, \ldots, E_8$ are disjoint copies of $\mathbf{P}^1$.

Let $\mathbb{L}_n$ be the lattice generated by n and by the classes of $E_1 \ldots E_8$. As an abstract lattice $\mathbb{L}_n$ is known as the *Nikulin lattice*.

Definition 5.14 A K3 surface (S, L) of genus g is a K3 quotient by a symplectic involution if it is endowed with a Nikulin class n such that:

- $i : \mathbb{L}_n \to Pic\ S$ is a primitive embedding,
- n is orthogonal to $c = c_1(L)$.

The moduli spaces of K3 quotients by a symplectic involution are known. Their irreducible components have dimension 11 and are classified for every g, see [SvG]. In these notes we are interested to the most natural irreducible component, which is birationally defined for every g as follows.

Definition 5.15 $\mathcal{F}_g^N$ is the closure in $\mathcal{F}_g$ of the moduli of pairs (S, L) such that $Pic(S) = \mathbb{Z}c \oplus \mathbb{L}_n$, where $c = c_1(L)$, n is a Nikulin class and $c \cdot n = 0$.

It is well known that $\mathcal{F}^N$ is integral of dimension 11. It is an irreducible component of the moduli of K3 surfaces of genus g which are K3 quotients by a symplectic involution. In these notes we fix the following definition.

Definition 5.16 A Nikulin surface of genus g is a K3 surface (S, L) of genus g with moduli point in $\mathcal{F}_g^N$.

For a general Nikulin surface (S, L) the morphism $f : S \to \mathbf{P}^g$, defined by $L = \mathcal{O}_S(C)$, factors as $f = \overline{f} \circ \sigma$. Here $\sigma : S \to \overline{S}$ is the contraction of $E_1, \ldots, E_8$ already considered and $\overline{f} : \overline{S} \to \mathbf{P}^g$ is an embedding.

In what follows we assume $\overline{S} \subset \mathbf{P}^g$ via the embedding $\overline{f}$. In particular the hyperplane sections of $\overline{S}$ are the canonical models $\overline{C}$ of the previously considered Prym canonical curves $C \subset S \subset \mathbf{P}^{g-2}$.

The next table shows some analogies which are unexpected. They occur between the sequence of families of hyperplane sections $\overline{C}$ of Nikulin surfaces $\overline{S}$ of genus g and the sequence of the families of hyperplane sections of general K3 surfaces of genus g. See the next section and [FV3].

- Parameter count. *C cannot be general for $g \geq 8$. On the other hand one expects the opposite for $g \leq 7$.*
- Genus 6 is unexpected. *$\overline{C}$ is a linear section of a particular quasi homogeneous space. This makes it not general.*
- Genus $g \leq 5$ and $g = 7$. *As expected $\overline{C}$ has general moduli.*
- $g = 7$. *A general Prym canonical curve C admits a unique embedding in a Nikulin surface S.*

- $g = 7$. *This defines a morphism* $\mathcal{R}_7 \to \mathcal{F}_7^N$ *which is a projective bundle over a non empty open subset of* $\mathcal{F}_7^N$.
- $g = 6$. C *has special syzygies, indeed it is not quadratically normal.*

5.4 Nikulin Surfaces and $\mathcal{R}_7$

To discuss some of the predicted analogies we start from genus 7. This, by dimension count, is the biggest value of g where one can have embeddings $C \subset S \subset \mathbf{P}^{g-2}$ so that C is a general Prym canonical curve and S is a Nikulin surface. The analogy here is with the family of general K3 surfaces of genus 11, the last value of g such that the family of hyperplane sections of K3 surfaces of genus g dominates $\mathcal{M}_g$. In genus 11 we have a map

$$f : \mathcal{M}_{11} \to \mathcal{F}_{11},$$

constructed by Mukai. In genus 7, we will show that there exists a quite simple analogous map

$$f^N : \mathcal{R}_7 \to \mathcal{F}_7^N.$$

In both cases the fibre of the map at the moduli point of $(S, \mathcal{O}_S(C))$ is the image in $\mathcal{R}_7$ of the open subset of $|C|$ parametrizing smooth curves.

Let us define f^N. Assume (C, η) is a general Prym curve of genus $g = 7$. We recall that then $\omega_C \otimes \eta$ is very ample and that the Prym canonical embedding $C \subset \mathbb{P}^5$ defined by $\omega_C \otimes \eta$ is projectively normal.

Note that we have $h^0(\mathcal{I}_C(2)) = 3$ and $C \subset S \subset \mathbf{P}^5$, where S is the base scheme of the net of quadrics $|\mathcal{I}_C(2)|$.

Lemma 5.17 *S is a smooth complete intersection of three quadrics.*

The proof of this lemma is shown in [FV3], as well as the next arguments reported here. The lemma implies that S is a smooth K3 surface. Due to Theorem 5.11, it will be not surprising the following result:

Lemma 5.18 $(S, \mathcal{O}_S(C))$ *is a Nikulin surface of genus* 7.

The construction uniquely associates a Nikulin surface S to the Prym curve (C, η). Therefore it defines a rational map

$$f^N : \mathcal{R}_7 \to \mathcal{F}_7^N.$$

Recall that $\mathcal{F}_g^N$ is integral of dimension 11 and notice that f^N is dominant. Moreover the fibre of f^N, at the moduli point of a general Nikulin surface S as above, is the family of smooth elements of $|C|$. Then it is not difficult to conclude that a birational model of $\overline{\mathcal{R}}_7$ is a $\mathbf{P}^7$-bundle over $\mathcal{F}_7^N$. This argument immediately implies the unirationality of $\mathcal{R}_7$, once we have shown that $\mathcal{F}_7^N$ is unirational. Let us sketch the proof of this property, see [FV3].

Theorem 5.19 *$\mathcal{F}_7^N$ is unirational.*

Proof With the previous notation let $C \subset S \subset \mathbf{P}^5$, where C is Prym canonical and S a general Nikulin surface. Let $H \in |\mathcal{O}_S(1)|$, as we have seen in proposition 5.13 S contains 8 disjoint lines $E_1 \dots E_8$ such that $E_1 + \dots + E_8 \sim 2C - 2H$ and $E_1C = \dots = E_8C = 0$. Fixing the line E_8 we consider the curve

$$R \in | \, C - E_1 - \dots - E_7 \, | \, .$$

Note that $R^2 = -2$ and $HR = 5 \geq 0$, then R exists and it is an isolated -2 curve of degree 5. Since S is general R is a rational normal quintic in $\mathbf{P}^5$. Now fix a rational normal quintic $R \subset \mathbf{P}^5$ and consider a general point $x := (x_1, \dots, x_{14}) \in R^{14}$. We can consider the curve

$$C_0 := R \cup \overline{x_1x_8} \cup \dots \cup \overline{x_7x_{14}} \subset \mathbf{P}^5.$$

C_0 is union of R and seven disjoint lines $\overline{x_ix_{i+7}}$, $i = 1 \dots 7$. One can show that C_0 is contained in a smooth complete intersection S of three quadrics. S is actually a Nikulin surface. To see this consider in S the seven lines $E_i := \overline{x_ix_{i+7}}$, $i = 1 \dots 7$ and $H \in |\mathcal{O}_S(1)|$. Observe that there exists one line more, namely

$$E_8 \sim 2C_0 - 2H - E_1 - \dots - E_7 = 2R + E_1 + \dots + E_7 - 2H.$$

Hence x uniquely defines a Nikulin surface S and we have constructed a dominant rational map $R^{14} \to \mathcal{F}_7^N$. $\square$

The previous theorem implies that

Theorem 5.20 *$\mathcal{R}_7$ is unirational.*

One can do better, see [FV6, Theorem 1.3]: let $\tilde{\mathcal{F}}_g^N$ be the moduli space of Nikulin surfaces of genus 7 endowed with one of the lines $E_1, \dots, E_8$. Then:

Theorem 5.21 *$\tilde{\mathcal{F}}_7^N$ is rational.*

6 Unirationality and $\mathcal{M}_g$

6.1 The Program for g ≤ 14

We started these notes with a discussion on the rationality problem for $\mathcal{M}_g$ in very low genus. In this last section we complement it by a short discussion on the known unirationality / uniruledness results for $\mathcal{M}_g$.

Up to now this means $g \leq 16$: the uniruledness of $\mathcal{M}_g$ is known for $g = 16$, after [CR3] and [BDPP]: see [F]. The rational connectedness is known for $g = 15$ [BrV], and the unirationality for $g \leq 14$, [S, CR1, Ser1, Ve1] for $g < 14$. We will mainly describe the constructions given in [Ve1] in order to prove the unirationality for $g \leq 14$. See also Schreyer's paper [Sc], where the methods in use are improved by the support of computational packages.

Let us review the program we intend to follow, even if it is not so surprising after reading the previous sections.

- Actually we are going to sketch the unirationality of some universal Brill Noether loci $\mathcal{W}^r_{d,g}$ dominating $\mathcal{M}_g$.
- We will use families of *canonical complete intersection surfaces* $S \subset \mathbf{P}^r$. The list of their types is short: (5), (3,3), (2,4), (2,2,3), (2,2,2,2).
- Let $C \subset S \subset \mathbf{P}^r$ be a smooth integral curve of genus g. We will study the possible cases where there exists *a linkage of* C *to a curve* B:

$$S \cdot F = C \cup B,$$

so that F is a hypersurface and B is a smooth integral of genus $p < g$. This will be useful to parametrize $\mathcal{M}_g$ by a family of curves of lower genus.

Now let $V \subset H^0(\mathcal{O}_B(1))$ be the space defining the embedding $B \subset \mathbf{P}^r$ and let $\mathcal{G}$ be the moduli space of fourtuples (B, V, S, F). Then the assignment $(B, V, S, F) \longrightarrow (C, \mathcal{O}_C(1))$, determined by $S \cdot F$, defines a rational map

$$\psi : \mathcal{G} \to \mathcal{W}^r_{d,g}.$$

On the other hand let $b = \deg B$, then we have the rational map

$$\phi : \mathcal{G} \to Pic_{b,p}$$

induced by the assignment $(B, V, S, F) \to (B' \mathcal{O}_B 1))$.

In the effective situations to be considered ψ is dominant and $\mathcal{G}$ birational to $Pic_{b,p} \times \mathbf{P}^n$ for some n. Moreover, for $p \leq 9$, $Pic_{b,p}$ is unirational, therefore $\mathcal{G}$ and also $\mathcal{W}^r_{d,g}$ are unirational. An outcome of this program is represented by the next theorem [Ve1].

Theorem 6.1

(1) *genus 14:* $\mathcal{W}^1_{8,14}$ *is birational to* $Pic_{14,8} \times \mathbf{P}^{10}$,
(2) *genus 13:* $\mathcal{W}^2_{11,13}$ *is dominated by* $Pic_{12,8} \times \mathbf{P}^8$,
(3) *genus 12:* $\mathcal{W}^0_{5,12}$ *is birational to* $Pic_{15,9} \times \mathbf{P}^5$,
(4) *genus 11:* $\mathcal{W}^0_{6,11}$ *is birational to* $Pic_{13,9} \times \mathbf{P}^3$.

The previous Brill-Noether loci dominate their corresponding spaces $\mathcal{M}_g$ via the forgetful map. So the next corollary is immediate.

Corollary 6.2 *$\mathcal{M}_g$ is unirational for $g = 11, 12, 13, 14$.*

In the next sections we briefly outline the required ad hoc constructions for the proofs of these results.

6.2 The Case of Genus 14

Let (C, L) be a general pair such that $g = 14$ and $L \in W^1_8(C)$. It is easy to see that $\omega_C(-L)$ is very ample and defines an embedding $C \subset \mathbf{P}^6$.

Proposition 6.3 *Let $\mathcal{I}_C$ be the ideal sheaf of C:*

- *C is projectively normal, in particular $h^0(\mathcal{I}_C(2)) = 5$.*
- *A smooth complete intersection of 4 quadrics contains C.*
- *C is linked to a projectively normal integral curve B of genus 8 by a complete intersection of 5 quadrics.*
- *$B \cup C$ is a nodal curve.*

See [Ve1, Sect. 4]. In particular B is smooth of degree 14. Let $\mathcal{H}$ be the Hilbert scheme of B in $\mathbf{P}^6$. The previous properties are satisfied in an irreducible neighborhood $\mathcal{U} \subset \mathcal{H}$ of B which is $Aut\ \mathbf{P}^6$-invariant. Here we have that $\mathcal{U}/Aut\ \mathbf{P}^6$ is birational to $Pic_{14,8}$ via the natural moduli map.

Let $D \in \mathcal{U}$ and let $\mathcal{I}_D$ be its ideal sheaf. We can assume that D is projectively normal so that $h^0(\mathcal{I}_D(2)) = 7$. Then, over a non empty open set of $Pic_{14,8}$, let us consider the Grassmann bundle

$$\phi : \mathcal{G} \to Pic_{14,8}$$

with fibre $G(5, H^0(\mathcal{I}_D(2)))$ at the moduli point of $(D, \mathcal{O}_D(1))$. Note that, counting dimensions, $\dim\ \mathcal{G} = \dim\ \mathcal{W}^1_{8,14} = \dim\ \mathcal{M}_{14}$. Using the linkage of C and B we can finally define a rational map

$$\psi : \mathcal{G} \to \mathcal{W}^1_{8,14}.$$

Let $(D, \mathcal{O}_D(1), V)$ be a triple defining a general point $x \in \mathcal{G}$. Then V is a general subspace of dimension 5 in $H^0(\mathcal{I}_D(2))$. Let $|V|$ be the linear system of quadrics defined by V. Then its base scheme is a nodal curve $D \cup C'$ such that the pair $(C', \omega_{C'}(-1))$ defines a point $y \in \mathcal{W}^1_{8,14}$. This follows because $\mathcal{G}$ contains the moduli point of the triple $(B, \mathcal{O}_B(1), H^0(\mathcal{I}_C(2)))$ and the same open property holds for the pair $(C, \omega_C(-1))$ defined by this triple. Then we set by definition $\psi(x) = y$. The map ψ is clearly invertible: to construct ψ^{-1} just observe that $V = H^0(\mathcal{I}_{C'}(2))$. We conclude that

$$\mathcal{W}^1_{8,14} \cong Pic_{14,8} \times G(5, 7).$$

Remark 6.4 (Plane Octics of Genus 8 with Two Triple Points) To guarantee the construction one has to prove that a general curve B is generated by quadrics and projectively normal. The proof works well by a computational package or else geometrically, [Sc] and [Ve1]. The geometric way in [Ve1] relies on a family of rational surfaces $X \subset \mathbf{P}^6$ containing smooth curves B of genus 8 with the required properties. Studying more properties of the curves B of this family has some interest. We study such a family here, so going back to singular plane curves. Indeed these curves B admit a simple plane model of degree 8 of special type and are general in moduli.

A general X is a projectively normal surface of degree 10 and sectional genus 5. Moreover it is the blowing up $\sigma : X \to \mathbf{P}^2$ of 11 general points. Let $L_1 \dots L_5, E_1 \dots E_6$ be the exceptional lines of σ, $P \in |\sigma^* \mathcal{O}_{\mathbf{P}^2}(1)|$, $H \in |\mathcal{O}_X(1)|$. It turns out that

$$|H| := |6P - 2(L_1 + \cdots + L_5) - (E_1 + \cdots + E_6)|$$

and that $|2P - L_1 - L_2 - E_1 - E_2|$ is a base point free pencil of rational normal sextics R. The curves B are then elements of $|2H - R|$ that is

$$|10P - 3(L_1 + L_2) - 4(L_3 + L_4 + L_5) - (E_1 + E_2) - 2(E_3 + \cdots + E_6)|,$$

see [Ve1]. Notice that in $\mathbf{P}^2$ the curve $\sigma(B)$ has three 4-tuples points $f_i = \sigma(L_i)$, $(i = 3, 4, 5)$, two triple points $t_j = \sigma(L_j)$, $(j = 1, 2)$, and four nodes $n_k = \sigma(E_k)$, $(k = 3, 4, 5, 6)$. Then consider the quadratic transformation

$$q : \mathbf{P}^2 \to \mathbf{P}^2,$$

centered at f_3, f_4, f_5 and the strict transform Γ of $\sigma(B)$. Then Γ is a general octic curve with seven nodes and two triple points. Let us see that the family of curves Γ dominates $\mathcal{M}_8$ and admits the following description.

In the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ consider the Severi variety $\mathcal{V}$ of nodal integral curves Γ' of type $(5, 5)$ and genus 8. Note that a general curve Γ is obtained by projecting some Γ' from one of its nodes. Moreover a general curve D of genus 8 is birational to some Γ': to see this just take two distinct $L_1, L_2 \in W^1_5(D)$. These define a generically injective morphism $D \to \mathbf{P}^1 \times \mathbf{P}^1$ whose image belongs to $\mathcal{V}$.

6.3 Genus 11, 12, 13

We give a brief description of the constructions used in genus 11, 12 and 13, see [Sc, Ve1] for details.

- *Genus 11.*

  Let (C, L) be a pair defining a general point of $\mathcal{W}^0_{6,11}$, then $\omega_C(-L)$ is a very ample line bundle. It defines an embedding $C \subset \mathbf{P}^4$ as a projectively normal curve of degree 14. The analogy to the situation described in genus 14 is complete. Let us summarize:

C is linked to a smooth integral curve B of degree 13 and genus 9 by a complete intersection of 3 cubics. B is projectively normal and $B \cup C$ is nodal. Let $\mathcal{H}$ be the Hilbert scheme of B. Then B has an irreducible neighborhood $\mathcal{U} \subset \mathcal{H}$ such that $\mathcal{U}/Aut\, \mathbf{P}^4$ is birational to $Pic_{13,9}$.

Let $p \in Pic_{13,9}$ be a general point representing the pair (D, L), then D is embedded in $\mathbf{P}^4$ by L. Let $\mathcal{I}_D$ be its ideal sheaf then $h^0(\mathcal{I}_D(3)) = 4$. Therefore we have a $\mathbf{P}^3$-bundle $\phi : \mathcal{G} \to Pic_{13,9}$ with fibre $|\mathcal{I}_D(3)|^*$ at p. As in genus 14 we can use linkage of curves to define a map

$$\psi : \mathcal{G} \to \mathcal{W}^0_{6,11}.$$

With the same arguments as in genus 14, we show that ψ is birational.

- *Genus 12*

In this case we can go back to nodal curves $C \cup B$ which are complete intersection of five quadrics in $\mathbf{P}^6$. The situation does not change:

C is general of degree 17 and genus 12 in $\mathbf{P}^6$ and the pair $(C, \omega_C(-1))$ defines a point of $\mathcal{W}^0_{5,12}$. B is general of degree 15 and genus 9 in $\mathbf{P}^6$ and $(B, \mathcal{O}_B(1))$ defines a point $p \in Pic_{15,9}$. We have $h^0(\mathcal{I}_B(2)) = 6$. Then we have a $\mathbf{P}^5$-bundle $\phi : \mathcal{G} \to Pic_{15,9}$ with fibre $|\mathcal{I}_B(2)|^*$ at p. As in the previous cases the linkage of B and C defines a birational map $\psi : \mathcal{G} \to \mathcal{W}^0_{5,12}$.

- *Genus 13*

Here we need a different construction. We consider the Severi variety $\mathcal{V}_{11,13}$ of plane curves of degree 11 and genus 13. $\mathcal{V}_{11,13}$ dominates the universal Brill-Noether locus $\mathcal{W}^2_{11,13}$ and this dominates $\mathcal{M}_{13}$. Let

$$\tilde{\mathcal{W}}^2_{11,13}$$

be the moduli space of triples (C, L, n) such that $[C, L] \in \mathcal{W}^2_{11,13}$, $\Gamma := f_L(C)$ is nodal and $f_L(n) \in Sing\, \Gamma$. Now we construct a birational map

$$\psi : Pic_{12,8} \times \mathbf{P}^{12} \to \tilde{\mathcal{W}}^2_{11,13}.$$

The existence of ψ implies the unirationality of $\mathcal{W}_{11,13}$ and of $\mathcal{M}_{13}$.

We start with triples (B, M, o) such that (B, M) defines a general point of $Pic_{12,8}$ and $o \in \mathbf{P}^4 := \mathbf{P}H^0(M)^*$ is general. We denote their moduli space by $\mathcal{P}$. Clearly $\mathcal{P}$ is birational to a $\mathbf{P}^4$-bundle on $Pic_{12,8}$. Let x be the moduli point of (B, M, o), if x is general we can assume that B is embedded by M in $\mathbf{P}^4$ as a projectively normal curve and that B is generated by cubics.

We have $h^0(\mathcal{I}_B(3)) = 6$. Then, for a general triple (B, M, o), one can show that there exists a unique cubic $F_o \in |\mathcal{I}_B(3)|$ such that $Sing\, F_o = \{o\}$ and o is a node for it. Let $\mathcal{J}_x$ be the ideal sheaf of $B \cup \{o\}$ in F_o, we consider the Grassmannian $\mathcal{G}_x := G(2, H^0(\mathcal{J}_x(3))$ and then the Grassmann bundle

$$\phi : \mathcal{G} \to \mathcal{P}$$

with fibre $\mathcal{G}_x$ at x. Since $\dim \mathcal{G}_x = 8$ and $\mathcal{P} \cong Pic_{12,8} \times \mathbf{P}^4$, it follows that $\mathcal{G}$ is birational to $Pic_{12,8} \times \mathbf{P}^{12}$. One defines a rational map

$$\psi : \mathcal{G} \to \tilde{\mathcal{W}}^2_{11,13}$$

as follows. A general $p \in \mathcal{G}_x \subset \mathcal{G}$ defines a general pencil P of cubic sections of F_o through $B \cup \{o\}$. One can show that the base scheme of P is a nodal curve $B \cup C_n$ where C_n is integral of genus 13 and degree 15, singular exactly at o. Let $\nu : C \to C_n$ be the normalization map and let $n = \nu^* o$. Putting

$$L := \omega_C(n) \otimes \nu^* \mathcal{O}_{C_n}(-1)$$

one can check that $L \in W^2_{11}(C)$ and that $h^0(L(-n)) = 2$. Hence the triple (C, L, n) defines a point $y \in \tilde{\mathcal{W}}_{11,13}$ and we set $\psi(p) = y$. Let us just mention how to invert ψ: starting from (C, L, n) one reconstructs the curve $C_n \subset \mathbf{P}^4$ from the line bundle $\omega_C(n) \otimes L^{-1}$. Let $\mathcal{I}_{C_n}$ be its ideal sheaf, it turns out that $h^0(\mathcal{I}_{C_n}(3)) = 3$. The base scheme of $|\mathcal{I}_{C_n}(3)|$ is $B \cup C_n$.

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Unirationality of Moduli Spaces of Special Cubic Fourfolds and K3 Surfaces

Howard Nuer

Abstract We provide explicit descriptions of the generic members of Hassett's divisors $\mathcal{C}_d$ for relevant $18 \leq d \leq 38$ and for $d = 44$, which furthermore gives unirationality of these $\mathcal{C}_d$. It follows as a corollary that the moduli space $\mathcal{N}_d$ of polarized K3 surfaces of degree d is unirational for $d = 14, 26, 38$. The case $d = 26$ is entirely new, while the other two cases have been previously proven by Mukai. We also explain the construction of what we conjecture to be a new family of hyperkähler manifolds which are not birational to any moduli space of (twisted) sheaves on a K3 surface.

This note is the summary of a lecture, based on the paper [Nue15], which the author gave at the summer school "Rationality problems in algebraic geometry" organized by CIME-CIRM in Levico Terme in June 2015. He would like to thank Rita Pardini and Pietro Pirola for affording him the honor of speaking and for fostering such a productive atmosphere.

1 Introduction

In [Nue15], we systematically provide concrete descriptions of special cubic fourfolds of discriminant $d \leq 44, d \neq 42$, recovering descriptions of previously known cases for $d = 12, 14, 20$. Recall that a smooth cubic fourfold $X \subset \mathbb{P}^5$, the vanishing locus of a degree 3 homogeneous polynomial in 6 variables, is called *special* if X contains the class of an algebraic surface S not homologous to a complete intersection. In other words, X is special if and only if the group $A^2(X)$ of codimension 2 cycles has rank at least 2. Let us denote by h^2 the square of the hyperplane class. Hassett defined in [Has00] the locus $\mathcal{C}_d$ of cubic fourfolds X which

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contain a positive-definite saturated rank two sublattice K of discriminant d which contains h^2. Hassett proved the following fundamental result:

Theorem 1.1 ([Has00, Theorem 1.0.1]) *$\mathcal{C}_d$ is an irreducible algebraic divisor in the twenty-dimensional moduli space $\mathcal{C}$ of smooth cubic fourfolds, and every special cubic fourfold is contained in some $\mathcal{C}_d$. Moreover, $\mathcal{C}_d \neq \varnothing$ if and only if $d > 6$ is an integer with $d \equiv 0, 2 \pmod 6$.*

For the almost 20 years since Hassett's work, the only $\mathcal{C}_d$ whose generic member X could be described explicitly were for $d = 8, 12, 14$, and 20. The surface S in these cases were given by planes, cubic scrolls, quintic del Pezzos, and Veronese surfaces, respectively. For some of these choices of d, the generic $X \in \mathcal{C}_d$ admits an alternative description. For example, the generic $X \in \mathcal{C}_8$ can be described as containing an octic K3 surface, and the generic $X \in \mathcal{C}_{14}$ contains a quartic scroll. It is nevertheless notable that for each d above, the surface S can be taken to be a smooth rational surface.

2 Explicit Descriptions of the Divisors $\mathcal{C}_d$ for $d \leq 38$ and $d = 44$

To obtain our results we begin with a fixed smooth surface S_0 and a smooth cubic fourfold X_0 containing it. Consider the flag Hilbert scheme

$$\mathbf{FH} := \{(S, X) | S \subset X\} \subset \mathrm{Hilb}^{S_0}_{\mathbb{P}^5} \times V,$$

where $V \subset \mathbb{P}^{55} = \mathrm{Hilb}^{X_0}_{\mathbb{P}^5}$ is the open set parametrizing smooth cubic fourfolds.[1] Then $\mathbf{FH}$ parametrizes flags of surfaces $S \subset X$, where S is an embedded deformation of S_0 and X is a smooth cubic fourfold, and comes with the following projections

$$\begin{array}{ccc} \mathbf{FH} & \xrightarrow{q_2} & V \\ {\scriptstyle q_1}\big\downarrow & & \\ \mathrm{Hilb}^{S_0}_{\mathbb{P}^5} & & \end{array}.$$

Note that $q_1^{-1}(S) = \mathbb{P}(H^0(\mathcal{I}_{S/\mathbb{P}^5}(3))) \cap V$ and $q_2^{-1}(X) = \mathrm{Hilb}^S_X$. By using semi-continuity, the deformation theory of Hilbert schemes of flags, and appropriately chosen surfaces S_0, we reduce the proof that $\dim q_2(\mathbf{FH}) = 54$, and thus that $\mathbf{FH}$

[1]The moduli space of smooth cubic fourfolds $\mathcal{C}$ is obtained as the GIT quotient $V // \mathrm{PGL}_6$.

Table 1 Smooth rational surfaces

d	n	H	H^2	$H.K_S$
12	7	$4L-(E_1+\ldots+E_6)-3E_7$	6	-4
12	13	$5L-(E_1+\ldots+E_{12})-2E_{13}$	9	-1
12	16	$7L-(E_1+\ldots+E_9)-2(E_{10}+\ldots+E_{16})$	12	2
14	4	$3L-E_1-E_2-E_3-E_4$	5	-5
14	9	$4L-E_1-\ldots-E_9$	7	-3
14	11	$5L-(E_1+\ldots+E_9)-2(E_{10}+E_{11})$	8	-2
14	14	$6L-(E_1+\ldots+E_{10})-2(E_{11}+\ldots+E_{14})$	10	0
14	15	$7L-(E_1+\ldots+E_9)-2(E_{10}+\ldots+E_{14})-3E_{15}$	11	1
14	16	$8L-(E_1+\ldots+E_6)-2(E_7+\ldots+E_{15})-3E_{16}$	13	3
18	12	$6L-(E_1+\ldots+E_7)-2(E_8+\ldots+E_{12})$	9	-1
18	15	$8L-(E_1+\ldots+E_6)-2(E_7+\ldots+E_{13})-3(E_{14}+E_{15})$	12	2
20	0	$2L$	4	-6
20	10	$6L-(E_1+\ldots+E_4)-2(E_5+\ldots+E_{10})$	8	-2
20	13	$7L-(E_1+\ldots+E_6)-2(E_7+\ldots+E_{12})-3E_{13}$	10	0
20	14	$7L-(E_1+\ldots+E_6)-2(E_7+\ldots+E_{14})$	11	1
20	15	$8L-(E_1+E_2+E_3)-2(E_4+\ldots+E_{15})$	13	3
24	11	$7L-(E_1+\ldots+E_3)-2(E_4+\ldots+E_{10})-3E_{11}$	9	-1
24	14	$8L-(E_1+E_2+E_3)-2(E_4+\ldots+E_{13})-3E_{14}$	12	2
26	12	$7L-(E_1+E_2+E_3)-2(E_4+\ldots+E_{12})$	10	0
26	13	$8L-(E_1+E_2+E_3)-2(E_4+\ldots+E_{11})-3(E_{12}+E_{13})$	11	1
30	10	$7L-2(E_1+\ldots+E_{10})$	9	-1
32	11	$9L-E_1-2(E_2+\ldots+E_5)-3(E_6+\ldots+E_{11})$	10	0
36	12	$10L-2(E_1+\ldots+E_4)-3(E_5+\ldots+E_{12})$	12	2
38	10	$10L-3(E_1+\ldots+E_{10})$	10	0
38	11	$10L-2(E_1+E_2)-3(E_3+\ldots+E_{11})$	11	1

dominates $\mathcal{C}_d$ in the GIT quotient,[2] to a *Macaulay2* [GS] calculation. Specifically, we obtain the following result:

Theorem 2.1 *The generic element of $\mathcal{C}_d$ for $12 \le d \le 38$ contains a smooth rational surface obtained as the blow-up of $\mathbb{P}^2$ at n generic points and embedded into $\mathbb{P}^5$ via the very ample linear system $|H| = |aL-(E_1+\ldots+E_i)-2(E_{i+1}+\ldots+E_{i+j})-3(E_{i+j+1}+\ldots+E_n)|$, where H is given by Table 1. Moreover, the generic $X \in \mathcal{C}_{44}$ contains a Fano embedded Enriques surface (see [DM] for the definition of a Fano Enriques surface).*

[2] Of course, d here depends on the specific S_0 and follows according to formula (1) in [Nue15].

3 Unirationality of Some $\mathcal{C}_d$

It is often important to understand the geometry of moduli spaces themselves. In the case of the $\mathcal{C}_d$, this is often intimately linked to the type of surfaces contained in the generic $X \in \mathcal{C}_d$. Indeed, $\mathcal{C}_d$ is known to be unirational for $d = 8, 12, 14, 20$ precisely because the cubics they parametrize contain specific surfaces. For example, $\mathcal{C}_8$, which parametrizes cubic fourfolds containing a plane, can be seen to be unirational by noting that $X \in \mathcal{C}_8$ is defined up to scaling by a cubic equation of the form $f(x_0, \dots, x_5) = \sum_{i=1}^{3} Q_i(x_0, \dots, x_5)L_i(x_0, \dots, x_5)$, where the Q_i are quadrics and the linear forms L_i cut out the plane. Similarly, the geometry in the proof of Theorem 2.1 above gives unirationality as a consequence:

Theorem 3.1 *For* $12 \leq d \leq 44$*,* $d \neq 42$*,* $\mathcal{C}_d$ *is unirational.*

This is immediate for $d \neq 44$ from the dependence of the construction on a choice of points in $\mathbb{P}^2$ and follows from an interesting old result of Verra in case $d = 44$ [Ver84].

As unirationality of a moduli space indicates that the generic element can be written down explicitly in free coordinates, we hope that Theorem 3.1 will be helpful for further study, in particular with regard to studying the rationality of generic $X \in \mathcal{C}_d$.

One expects that $\mathcal{C}_d$ ceases to be unirational as d grows, and it is natural to ask what is the smallest d such that $\mathcal{C}_d$ is not unirational, and at the other extreme, one can ask if there is a minimal d after which $\mathcal{C}_d$ is of general type. Questions of this nature have been previously investigated by Gritsenko, Hulek, and Sankaran in the cases of polarized K3 surfaces and certain families of hyperkähler manifolds (see [GHS13] for a good account). They prove, for example, that the moduli space $\mathcal{N}_{2d}$ of polarized K3 surfaces of degree $2d$ has non-negative Kodaira dimension for $d \geq 40$, $d \neq 41, 44, 45, 47$, and is of general type for $d > 61$, as well as for $d = 46, 50, 52, 54, 57, 58, 60$. Hassett (see [Has00, Sect. 5.3]) demonstrated a surprising and beautiful connection between the period domains of cubic fourfolds and polarized K3 surfaces. In particular, he showed that for d not divisible by 4, 9, or any odd prime $p \equiv 2 \pmod 3$ there is a rational map $\mathcal{N}_d \dashrightarrow \mathcal{C}_d$ which is birational if $d \equiv 2 \pmod 6$ and a double cover if $d \equiv 0 \pmod 6$. Translating the result from [GHS13] about $\mathcal{N}_d$, we get the following:

Proposition 3.2 *Let* $d > 80, d \equiv 2 \pmod 6, 4 \nmid d$ *be such that for any odd prime* p*,* $p \mid d$ *implies* $p \equiv 1 \pmod 3$*. Then the Kodaira dimension of* $\mathcal{C}_d$ *is non-negative. If moreover* $d > 122$*, then* $\mathcal{C}_d$ *is of general type.*

Proposition 3.2 thus provides an infinite number of large d such that $\mathcal{C}_d$ is of general type, and one expects that the gaps can be filled in using automorphic form techniques as in the K3 case. This has recently been confirmed by Várilly-

Alvarado and Tanimoto [TVA].[3] The results of [TVA] bound from above the minimum discriminant required for $\mathcal{C}_d$ to have nonnegative Kodaira dimension, and Theorem 3.1 gives 44 as a lower bound.

4 Unirationality of $\mathcal{N}_d$ for $d = 14, 26, 38$

An interesting further consequence of Theorem 3.1 is that we can obtain new results about moduli spaces of K3 surfaces. By utilizing the aforementioned birational isomorphism of period domains from [Has00], we get the following:

Theorem 4.1 *The moduli space $\mathcal{N}_d$, parametrizing polarized K3 surfaces of degree d, is unirational for $d = 14, 26, 38$.*

The cases $d = 14, 38$ are already known due to the work of Mukai [Muk88, Muk92]. He obtains his results by demonstrating the generic $T \in \mathcal{N}_d$ as a complete intersection in a certain homogeneous space. The case $d = 26$ is entirely new as far we know and fills in a long-standing gap in known unirationality results ($\mathcal{N}_d$ was already known to be unirational for $2 \leq d \leq 24$ and $d = 30, 32, 34, 38$).[4] While this result proves the unirationality of $\mathcal{N}_{26}$, it does not provide a geometric construction of the generic K3 surface it parametrizes. Such a construction remains an interesting open problem.

5 Kuznetsov's Category and New Hyperkähler Manifolds

One of the most exciting and perplexing problems in the study of cubic fourfolds is determining whether or not they are rational. The generic cubic fourfold is expected to be irrational, but no known example of a cubic fourfold has been shown to be irrational. The most recent approach to this classical problem is due to Kuznetsov. He introduced in [Kuz10] the subcategory

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp}$$

of the bounded derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}(X)$ and proposed the following conjecture:

[3] In particular, they have shown that $\mathcal{C}_{6n+2}$ is of general type for $n > 18$ and $n \neq 20, 21, 25$ and has nonnegative Kodaira dimension for $n = 14, 18, 20, 21, 25$. Moreover, $\mathcal{C}_{6n}$ is of general type for $n = 19, 21, 24, 25, 26, 28, 29, 30, 31$ and $n \geq 34$, and it has nonnegative Kodaira dimension for $n = 17, 23, 27, 33$.

[4] $\mathcal{N}_{26}$ was, however, shown to have negative Kodaira dimension in A. Peterson's forthcoming thesis.

Conjecture 5.1 A smooth cubic fourfold X is rational if and only if $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(T)$ for a K3 surface T. When the latter condition holds, we say that $\mathcal{A}_X$ is geometric and T is associated to X in the categorical sense.

He furthermore verified his conjecture for all known rational cubic fourfolds. Work of Addington and Thomas [AT14] showed that Kuznetsov's condition on $\mathcal{A}_X$ is related to X having an associated K3 surface in the sense of Hassett (see [Has00, Definition 5.1.1]) in the following way:

Theorem 5.2 *If $\mathcal{A}_X$ is geometric, then $X \in \mathcal{C}_d$ for d not divisible by 4, 9, or any odd prime $p \equiv 2 \pmod 3$. Conversely, for such d, there is a Zariski open subset of $\mathcal{C}_d$ parametrizing those X such that $\mathcal{A}_X$ is geometric.*

One can easily show that in the range of cases considered above we have the following:

Proposition 5.3 *For $(d,n) = (14,14), (20,13), (26,12), (32,11), (38,10)$ and for $d = 44$, $\mathcal{I}_{S/X}(2) \in \mathcal{A}_X$ for generic S and X as above. Similarly, for $(d,n) = (20,0)$, $\mathcal{I}_{S/X}(1) \in \mathcal{A}_X$.*

For generic $X \in \mathcal{C}_d$, $d = 14, 26, 38$, $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(T)$ (resp. $X \cong \mathrm{D}^{\mathrm{b}}(T,\alpha)$ for $d = 32$) for some polarized K3 surface T by Theorem 5.2 (resp. [Huy15]), so for the corresponding p in the list from Proposition 5.3, $\mathcal{I}_{S/X}(2)$ can be thought of as an object of $\mathrm{D}^{\mathrm{b}}(T)$ (resp. $\mathrm{D}^{\mathrm{b}}(T,\alpha)$). There should be some Bridgeland stability condition $\sigma \in \mathrm{Stab}^{\dagger}(T)$ such that the Hilbert scheme $\mathrm{Hilb}_X^S \cong M_\sigma(v)$, where $M_\sigma(v)$ is the moduli space of σ-stable objects on T of an appropriate Mukai vector v, so that Hilb_X^S would then be a birational minimal model for a moduli space of stable sheaves on a K3 surface (see [BM12, BM13]). In all previously known cases, the K3 surface T could be observed geometrically in the construction of S and X. In a current work-in-progress, we are investigating the case $d = 38$, where the component of Hilb_X^S containing S is precisely the K3 surface T, and this can be seen from the projective geometry of the construction. It is an open question whether this connection can be used to prove rationality of the generic $X \in \mathcal{C}_{38}$.

Even for the remaining d considered in Proposition 5.3, the fact that $\mathcal{I}_{S/X}(2)$ or $\mathcal{I}_{S/X}(1) \in \mathcal{A}_X$ is interesting for another reason. Indeed, Kuznetsov and Markushevich have constructed in [KM09] a nondegenerate closed, holomorphic, symplectic form on the smooth part of any moduli space $\mathcal{M}$ parametrizing stable sheaves $\mathcal{F}$ with $\mathcal{F} \in \mathcal{A}_X$. Taking $\mathcal{M}$ to be the component of the Hilbert scheme Hilb_X containing S (but with universal sheaf $\mathcal{I}_{S/X}(2)$ instead), we get a holomorphic symplectic form on the smooth locus about S. For $d = 20, 44$, $\mathcal{A}_X \ncong \mathrm{D}^{\mathrm{b}}(T,\alpha)$, even for a nontrivial Brauer twist by α on the K3 surface T, by [Huy15]. In the case $(d,n) = (20,13)$ this Hilbert scheme should provide a new example of a holomorphic symplectic variety that is not birational to any moduli space of sheaves on a K3 surface (see [LLSvS] for the first such example). When $(d,n) = (20,0)$ or $d = 44$, we instead get new spherical objects, which have become very important in the study of derived categories. We are led to wonder what conditions on X are imposed by $\mathcal{A}_X$ having spherical objects unrelated to K3 geometry. We hope to return to both questions in forthcoming work.

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