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MATHEMATICS TIMES

CONCEPT OF THE MONTH

Properties of Parabola

CONCEPT EXPLORATION

Sets and Relations

CHALLENGING PROBLEMS (JEE)

SOLVED PAPER JEE MAIN - 2018

EVA AITS (JEE ADVANCED -2), (BITSAT -2)

MAY 2018

MATHEMATICS TIMES

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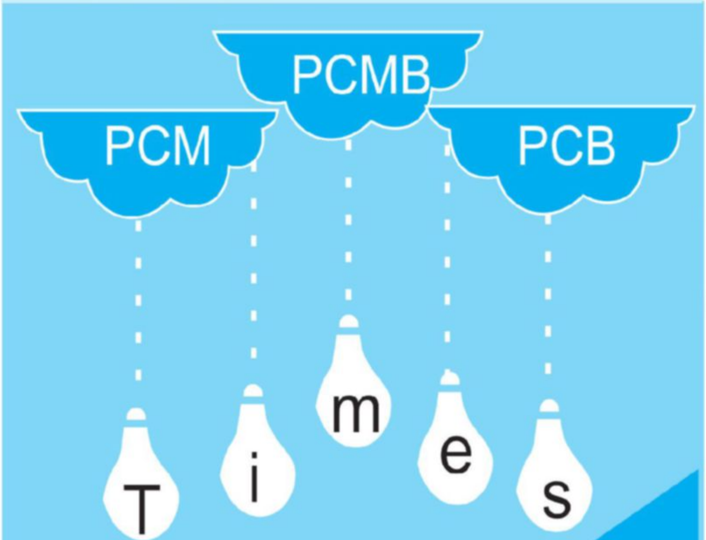
VOL-IV



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Properties of Parabola

Concept of the month

This column is aimed at preparing students for all competitive exams like JEE, BITSAT etc. Every concept has been designed by highly qualified faculty to cater to the needs of the students by discussing the most complicated and confusing concepts in Mathematics.

By. **SAWAN AGARWAL**(M.TECH)
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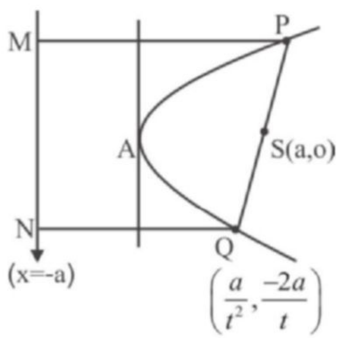
Parabola defination

A parabola is the Locus of a point which moves in a plane such that its distance from a fixed point (called focus) is always equal to its distance from a fixed straight line. (called directrix)

Properties of parabola

1. If the point $\left(at^2, 2at\right)$ be the extremity of a focal chord of parabola $y^2 = 4ax$, then the length of focal chord is $a\left(t+\frac{1}{t}\right)^2$

Proof: Since one extremity of focal chord is p $\left(at^2, 2at\right)$ then the other extremity is $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$



$\therefore$ The length of $PQ = SP + SQ$

$$= PM + QN = at^2 + a + \frac{a}{t^2} + a$$

$$= a\left[t^2 + \frac{1}{t^2} + 2\right]$$

$$= a\left[t + \frac{1}{t}\right]^2$$

2. Latus rectum is the smallest focal chord of any parabola

Proof: Using property -1

We know that length of focal chord is a $a\left(t+\frac{1}{t}\right)^2$

Now, $\left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0 \left(\because AM \geq GM\right)$

$$\therefore a\left(t + \frac{1}{t}\right)^2 \geq a(2)^2$$

$$\geq 4a$$

Length of focal chord $\geq$ Latus rectum.

3. Point of intersection of tangents at any two points t_1 and t_2 on the parabola will be $\left(at_1t_2, a\left(t_1+t_2\right)\right)$.

Proof: Let the parabola be $y^2 = 4ax$

Let the two points on the parabola are

$P=\left(at_1^2,2at_1\right)$ and $Q=\left(at_2^2,2at_2\right)$

Equation of tangents at P is $t_1y=x+at_1^2$,

and at Q is $t_2y=x+at_2^2$

solving these equations we get

$$x=at_1t_2,\ y=a\left(t_1+t_2\right)$$

Thus coordinates of point of intersection of tangents will be $\left(at_1t_2,a\left(t_1+t_2\right)\right)$

4. Locus of point of intersection of the mutually perpendicular tangents to a prabola is the directrix of the parabola

Proof: Let the points $P\left(at_1^2,2at_1\right)$ and $Q\left(at_2^2,2at_2\right)$ on the parabola $y^2=4ax$, tangents at P and Q are

$$t_1y=x+at_1^2 \qquad (1)$$

$$t_2y=x+at_2^2 \qquad (2)$$

and the point of intersection of these tangents is $\left(at_1t_2,a\left(t_1+t_2\right)\right)$, Let this point is (h,k)

i.e., $\quad h=at_1t_2,\ k=a\left(t_1+t_2\right)$

Slope of tangents (1) and (2) are $\frac{1}{t_1}$ and $\frac{1}{t_2}$

since the tangents are perpendiculars then

$$t_1t_2=-1.$$

Thus we get $h=-a, h+a=0$

$\therefore$ Locus of the point of intersection of tangents is $x+a=0$ which is directrix of $y^2=4ax$

5. If normals at 't_1' meets the parabola $y^2=4ax$ at some point 't_2' then $\left\{t_2=-t_1\frac{-2}{t_1}\right\}$.

proof: Let the parabola be $y^2=4ax$, equation of normal at $P\left(at_1^2,2at_1\right)$ is

$$y=-t_1x+2at_1+at_1^3$$

Since it meets the parabola again at $Q=\left(at_2^2,2at_2\right)$ then equation of normal i.e, $y=-t_1x+2at_1+at_1^3$ passes through $Q\left(at_2^2,2at_2\right)$ is

$$2at_2=-at_1t_2^2+2at_1+at_1^3$$

$\Rightarrow \quad 2a\left(t_2-t_1\right)+at_1\left(t_2^2-t_1^2\right)=0$

$\Rightarrow \quad a\left(t_2-t_1\right)\left(2+t_1\left(t_2+t_1\right)\right)=0$

i.e., $\quad 2+t_1\left(t_2+t_1\right)=0$

$$\therefore\ t_2=\frac{-2}{t_1}-t_1$$

6. If normals for the parabola $y^2=4ax$ at 't_1' and 't_2' meets the parabola at some point then $t_1t_2=2$

proof: Suppose normals meet at t_3 then

i.e., $\quad t_3=-t_1-\frac{2}{t_1}=-t_2-\frac{2}{t_2}$

$\Rightarrow \quad \left(t_1-t_2\right)=2\left(\frac{1}{t_2}-\frac{1}{t_1}\right)$

i.e., $\quad t_1t_2=2$

7. The algebraic sum of the slope of three concurrent normals is zero.

proof: Let $P(h,k)$ be any given point and $y^2=4ax$ be a parabola

The equation of any normal to $y^2=4ax$ is

$$y=mx-2am-am^3$$

It it passes through (h,k), then

$$k=mh-2am-am^3$$

$$\therefore\ am^3+m(2a-h)+k=0$$

This is a cubic equation in m, so that three roots say m_1,m_2 and m_3.

$$\therefore\ m_1+m_2+m_3=0,\ m_1m_2m_3=\frac{-k}{a}$$

and also $m_1m_2+m_2m_3+m_1m_3=\frac{(2a-h)}{a}$

$\therefore$ sum of slope=0

8. The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero

proof: Let the ordinates of A, B, C be y_1, y_2, y_3 respectively then.

$$y_1 = -2a\,m_1$$
$$y_2 = -2a\,m_2$$
$$y_3 = -2a\,m_3$$

algebraic sum of these ordinates is

$$y_1 + y_2 + y_3 = -2a(m_1 + m_2 + m_3) = 0$$

9. If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h,k) be real then $h > 2a$.

proof: When normals are real, then the three roots of equation $am^3 + m(2a-h) + k = 0$ are real and in this case.

$$m_1^{\,2} + m_2^{\,2} + m_3^{\,2} > 0$$

$$\Rightarrow (m_1 + m_2 + m_3)^2 - 2\,(m_1m_2 + m_2m_3 + m_1m_3) > 0$$

$$\Rightarrow \quad 0^2 - \frac{2(2a-h)}{a} > 0$$

$$\Rightarrow \quad h - 2a > 0$$

$$\Rightarrow \quad h > 2a$$

10. If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h,k) be real and distinct then $27ak^2 < 4(h-2a)^3$

Proof: Let $f(m) = am^3 + m(2a-h) + k$

now, $f'(m) = 3am^2 + (2a-h)$

Two distinct roots of $f'(m) = 0$ are

$$\alpha = \sqrt{\frac{h-2a}{3a}} \text{ and } \beta = -\sqrt{\frac{h-2a}{3a}}$$

Now, $\quad f(\alpha)f(\beta) < 0$

$$f(\alpha)f(-\alpha) < 0$$

on solving we get, $27ak^2 < 4(h-2a)^3$

SOLVED EXAMPLES

1. Show that the normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, -12). Find also the length of the normal chord.

Sol: $a = 2$, since normal at (x,y) to the parabola $y^2 = 4ax$ is, $y - y_1 = \frac{-y_1}{2a}(x - x_1)$, here $x_1 = 2$ and $y_1 = 4$ equation of normal is

$$y - 4 = \frac{-4}{4}(x-2) \Rightarrow x + y - 6 = 0 \qquad (1)$$

Solving (1) and $y^2 = 8x$

We get $y^2 = 8(6-y)$, $\therefore y = -12$ and $y = 4$,

then $\quad x = 18$ and $x = 2$

Hence proved, length of normal chord

$$PQ = \sqrt{(18-2)^2 + (12-4)^2} = 16\sqrt{2}$$

2. Find the point on the axis of the parabola $3y^2 + 4y - 6x + 8 = 0$ from where 3 distinct normals can be drawn

Sol: Given parabola is $3y^2 + 4y - 6x + 8 = 0$

$$3\left(y^2 + \frac{4}{3}y\right) = 6x - 8$$

$$\left(y + \frac{2}{3}\right)^2 = 2\left(x - \frac{10}{9}\right)$$

$$y + \frac{2}{3} = y,\ x - \frac{10}{9} = x$$

$$y^2 = 4\left(\frac{1}{2}\right)x;\ a = \frac{1}{2}$$

any point on the axis of parabola is $\left(x, \frac{-2}{3}\right)$

and $x > 2a \Rightarrow x - \frac{10}{9} > 1 \Rightarrow \frac{19}{9}$

3. A line PQ meets the parabola $y^2 = 4ax$ in R such that PQ is bisected at R. If the co-ordinates of P are (x_1, y_1), show that the locus of Q is the parabola $(y + y_1)^2 = 8a(x + x_1)$

Sol: Let the co-ordinates of Q is (h,k) since R is the mid point of PQ, (R lies on the parabola)

$$at^2 = \frac{x_1+h}{2}, t^2 = \frac{x_1+h}{2a}, 2at = \frac{y_1+k}{2}, \text{ and}$$

$$t = \frac{y_1+k}{4a}$$

from, using above equations, we have

$$\left(\frac{y_1+k}{4a}\right)^2 = \left(\frac{x_1+h}{2a}\right)$$

$$\Rightarrow \quad (y_1+k)^2 = 8a(x_1+h)$$

Hence locus of $Q(h,k)$ is

$$(y+y_1)^2 = 8a(x+x_1)$$

4. Prove that the locus of the point of intersection of the normals at the ends of a system of parallel chords of a parabola is a line which is normal to the given parabola

Sol: Let the parabola be $y^2 = 4ax$

Equation of normal at a point 't' is

$$y = -tx + 2at + at^3$$

Let $P(at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$

slope of PQ be $m = \dfrac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \dfrac{2}{t_1+t_2}$

Now normals at P and Q intersect at $R(x_1, y_1)$ then

$$x_1 = 2a + (t_1^2 + t_1t_2 + t_2^2), y_1 = at_1t_2(t_1+t_2).$$

$$\Rightarrow (x_1 - 2a) = a\left\{\frac{4}{m^2} - t_1t_2\right\}; y_1 = -at_1t_2\left(\frac{2}{m}\right)$$

$$\Rightarrow \quad y_1 = \left(\frac{2}{m}\right)x_1 - 2a\left(\frac{2}{m}\right) - a\left(\frac{2}{m}\right)^3$$

The locus of $R(x_1, y_1)$ is

$$y = -\left(\frac{-2}{m}\right)x + 2a\left(\frac{-2}{m}\right) + a\left(\frac{-2}{m}\right)^3$$

which is normal at the point whose parameter is $\dfrac{-2}{m}$

5. A parabola of latus rectum $4a$, touches a fixed parabola, the axes of the two curves being parallel; prove that the focus of the vertex of the moving curve is a parabola of $lr = 8a$.

Sol: Let the given parabola is $y^2 = 4ax$

If the vertex of moving parabola is (α, β) then equation of moving parabola is

$$(y-\beta)^2 = -4a(x-\alpha) \qquad (1)$$

Substituting the value of x, i,e $x = \dfrac{y^2}{4a}$ in eqn (1)

$$(y-\beta)^2 = -4a\left(\frac{y^2}{4a} - \alpha\right)$$

$$\Rightarrow 2y^2 - 2\beta y + \beta^2 - 4a\alpha = 0 \qquad (2)$$

since the two parabola $y^2 = 4ax$ and $(y-\beta)^2 = -4a(x-\alpha)$ touches each other hence roots of eqn (2) are equal.

i.e, $\quad D = 0$

$\Rightarrow \quad B^2 - 4AC = 0$

i.e., $\quad (-2\beta)^2 = 4\cdot 2(\beta^2 - 4a\alpha)$

$$4\beta^2 = 32a\alpha$$

$$\beta^2 = 8a\alpha$$

or, $y^2 = 8ax$, which is a parabola which has latus rectum $= 8a$.

6. TP and TQ are any two tangents to a parabola and the tangents at a third point R cuts them in P' and Q' prove that $\left(\dfrac{TP'}{TP} + \dfrac{TQ'}{TQ} = 1\right)$

Sol: Let parabola be $y^2 = 4ax$ and co-ordinates of P and Q on this parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

T is the point of intersection of tangents at t_1 and t_2.

$\therefore$ co-ordinates of $T \equiv (at_1t_2, a(t_1+t_2))$

Similarly $P' \equiv (at_3t_1, a(t_3+t_1))$

$$Q' \equiv \left(at_2t_3, a\left(t_2+t_3\right)\right)$$

Let $\quad TP' : TP = \lambda : 1$

$$\lambda = \frac{t_3-t_2}{t_1-t_2}, \frac{TP'}{TP} = \frac{t_3-t_2}{t_1-t_2}$$

Similarly, $\quad \dfrac{TQ'}{TQ} = \dfrac{t_1-t_3}{t_1-t_2}$

$$\frac{TP'}{TP} + \frac{TQ'}{TQ} = 1$$

7. Find the shortest distance between the parabola $y^2 = 4x$ and $y^2 = 2x-6$.

Sol: Shortest distance between two curves occurs along the common normal.

Normal to $y^2 = 4x$ at $\left(m^2, 2m\right)$ is

$y + mx - 2m - m^3 = 0.$

Normal to $y^2 = 2(x-3)$ at $\left(\dfrac{m^2}{2}+3, m\right)$ is

$y + mx - 4m - \dfrac{m^2}{2} = 0$ both are same if

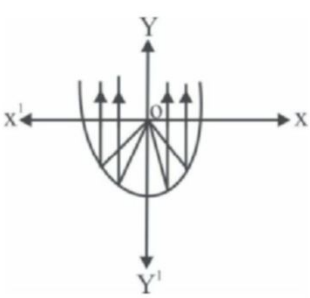
$-2m - m^3 = -4m - \dfrac{1}{2}m^3$

$\Rightarrow \qquad m = 0, \pm 2$

So, the points will be (4, 4) and (5, 2) or (4, -4) and (5, -2) Hence, shortest distance will be $\sqrt{1+4} = \sqrt{5}$.

8. The equation of the mirror that can reflect all incident rays from origin parallel to y-axis will be?

Sol: The equation of such mirror is an equation of parabola whose axis is y-axis and whose focus is (0, 0) required equation is $x^2 = 4a(y+a)$



9. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point (1, 2) is

Sol: Any point on the given parabola is $\left(t^2, 2t\right)$.

The equation of tangents at (1, 2) is $x - y + 1 = 0$

The image (h, k) of the point $\left(t^2, 2t\right)$ in $x - y + 1 = 0$ is given by,

$$\frac{h-t^2}{1} = \frac{k-2t}{-1} = \frac{-2\left(t^2-2t+1\right)}{1+1}$$

$\therefore h = 2t-1, k = t^2+1$

or, $(h+1)^2 = 4(k-1) \Rightarrow (x+1)^2 = 4(y-1)$

10. PQ is any focal chord of the parabola $y^2 = 32x$, The length of PQ can never be less than.

Sol: Length of focal chord is $a\left(t+\dfrac{1}{t}\right)^2$, if $\left(at^2, 2at\right)$ is one extremity of the parabola $y^2 = 4ax$.

$$t + \frac{1}{t} \geq 2\left(AM \geq GM\right)$$

$$\Rightarrow a\left(t+\frac{1}{t}\right)^2 \geq 4a \quad \text{or,} \quad PQ \geq 32$$

Exercise

1. The length of the latus-rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is **[2001]**

(a) 4 (b) 6 (c) 8 (d) 10

2. The equation of tangents to the parabola $y^2 = 4ax$ at the ends of its latus rectum is **[2001]**

(a) $x - y + a = 0$ (b) $x + y + a = 0$

(c) $x + y - a = 0$ (d) Both (a) and (b)

3. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then **[2004]**

(a) $d^2 + (2b+3c)^2 = 0$ (b) $d^2 + (3b+2c)^2 = 0$

(c) $d^2+(2b-3c)^2=0$ (d) $d^2+(3b-2c)^2=0$

4. The locus of the vertices of the family of parabolas $y=\frac{a^3x^2}{3}+\frac{a^2x}{x^2}-2a$ is **[2006]**

(a) $xy=\frac{3}{4}$ (b) $xy=\frac{35}{16}$

(c) $xy=\frac{64}{105}$ (d) $xy=\frac{105}{64}$

5. The equation of a tangents to the parabola $y^2=8x$ is $y=x+2$. The point on this line from which the tangents to the parabola is perpendicular to the given tangents is **[2007]**

(a) (-1, 1) (b) (0, 2) (c) (2, 4) (d) (-2, 0)

6. A parabola has the origin as its focus and the line $x=2$ as the directrix. Then the vertex of the parabola is at **[2008]**

(a) (1, 0) (b) (0, 1) (c) (2, 0) (d) (0,2)

7. If two tangents drawn from a point P to the parabola $y^2=4x$ are at right angles, then the locus of P is **[2010]**

(a) $x=1$ (b) $2x+1=0$

(c) $x=-1$ (d) $2x-1=0$

8. The shortest distance between line $y-x=1$ and curve $x=y^2$ is **[2011]**

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{3\sqrt{2}}{8}$ (c) $\frac{8}{3\sqrt{2}}$ (d) $\frac{4}{\sqrt{3}}$

9. The equation (S) of commom tangents (S) to the parabola $y=x^2$ and $y=-(x-2)^2$ **[2006]**

(a) $y=-4(x-1)$ (b) $y=0$

(c) $y=4(x-1)$ (d) $y=-30x-50$

10. **Statement-1:** The curve $y=-\frac{x^2}{2}+x+1$ is symmetric with respect to the line $x=1$ because
Statement-2: A parabola is symmetric about its axis. **[2007]**

(a) Statement-1 true, Statement-2 is true;statement-2 is a correct explanation for statement-1.

(b) Statement-1 true, statement-2 is true; statements-2 is not a correct explanation for statement-1

(c) Statement-1 is true, statement-2 is false

(d) Statement-1 is false, statement-2 is true

11. Consider the two curves **[2008]**

$C_1: y^2=4x$; $C_2: x^2+y^2-6x+1=0$ then,

(a) C_1 & C_2 touch each other only at one point

(b) C_1 & C_2 touch each other exactly at two points

(c) C_1 & C_2 intersect (but do not touch) at exactly two points

(d) C_1 and C_2 neither intersect nor touch each other

12. Let A and B be two distinct points on the parabola $y^2=4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be **[2010]**

(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$ (c) $\frac{2}{r}$ (d) $-\frac{2}{r}$

13. Let (x, y) be any point on the parabola $y^2=4x$. Let P be the point that divides the line segment from $(0,0)$ to (x,y) in the ratio 1:3. Then the locus of P is **[2011]**

(a) $x^2=y$ (b) $y^2=2x$

(c) $y^2=x$ (d) $x^2=2y$

14. Consider the parabola $y^2=8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2},2\right)$ on the parabola,and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[2011]**

(a) 2 (b) 3 (c) 4 (d) 5

ANSWER KEY

1. c	2. d	3. a	4. d
5. d	6. a	7. c	8. b
9. b,c	10. a	11. b	12. c
13. c	14. a		

HINTS & SOLUTIONS

1. Sol: $x^2-4x-8y+12=0$

$(x-2)^2=8y-8=8(y-1)$

$x^2=4(2)y$

Lenth of $LR=4a=8$

option c is correct

2.Sol:

$y^2=4ax$, ends of $LR\equiv(a,2a)$ and $(a,-2a)$

Equation of tangent

$$(y-2a)=\frac{2a}{2a}(x-a) \text{ and } (y+2a)=\left(\frac{2a}{-2a}\right)(x-a)$$

$$\Rightarrow y-2a=x-a \text{ and } y+2a=-1(x-a)$$

$$y-x-a=0 \text{ and } y+x+a=0$$

Option d is correct

3.Sol:

$$2bx+3cy+4d=0$$

$$y^2=4ax \text{ and } x^2=4ay$$

point of intersection (4a, 4a) and (0, 0)

$$\therefore 0+0+4d=0 \text{ and } 8ab+12ac+4d=0$$

$$d=0 \text{ and } 4a(2b+3c)=0 \text{ since } (a\neq 0)$$

$$\therefore 2b+3c=0$$

Option a is correct

4.Sol:

$$y=\frac{a^3x^2}{3}+\frac{a^2x}{2}-2a$$

$$=\frac{a^3}{3}\left[x^2+\frac{3}{2a}x\right]-2a$$

$$=\frac{a^3}{3}\left[\left(x+\frac{3}{4a}\right)^2-\frac{9}{16a^2}\right]-2a$$

$$=\frac{a^3}{3}\left[\left(x+\frac{3}{4a}\right)\right]^2-\frac{a^3\cdot 9}{3\cdot 16.a^2}-2a$$

$$y=\frac{a^3}{3}\left(x+\frac{3}{4a}\right)^2-\frac{3}{16}a-2a$$

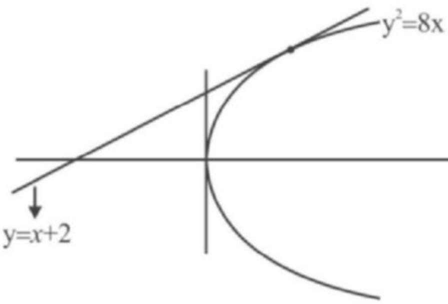
$$\left(y+\frac{35}{16}a\right)=\frac{a^3}{3}\left(x+\frac{3}{4a}\right)^2$$

Vertex $(h,k)=\left(\frac{-3}{4a},\frac{-35}{16}a\right)$

$$hk=\frac{105}{64},\quad xy=\frac{105}{64}$$

Option d is correct

5.Sol:



The point will be the point on directrix

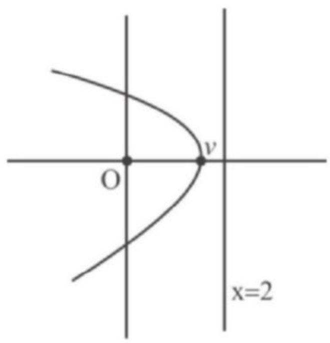
$\Rightarrow x=-2, y=0$

$\therefore$ point (-2, 0)

Option d is correct

6.Sol:

$V(1,0)$



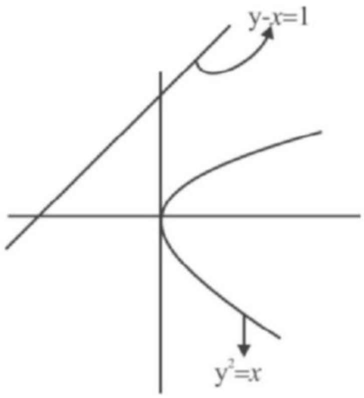
Option a is correct

7.Sol: Locus of P would be the directrix of $y^2=4x$

which is $x+1=0$

Option c is correct

8.Sol:



The shortest distance between two curves is always along the common normal

Slope of tangent parallel to $y-x=1$ is 1

$$\therefore \quad 2yy'=1$$

$$y'=\frac{1}{2y}=1 \Rightarrow y=\frac{1}{2}, x=\frac{1}{4}$$

Point on curve $\left(\frac{1}{4},\frac{1}{2}\right)$

$\therefore$ Equation of parallel tangent,

$$y-x=c, \quad \frac{1}{2}-\frac{1}{4}=c, \quad c=\frac{1}{4}$$

$$\therefore y-x=\frac{1}{4}$$

Distance between the two parallel lines

$$=\left|\frac{3/4}{\sqrt{2}}\right|=\frac{3\sqrt{2}}{8}$$

Option b is correct

9.Sol: $y=x^2$ and $y=-(x-2)^2$

$x^2=y$ and $(x-2)^2=-y$

Equation of tangent to $x^2=4ay$ is $y=mx-am^2$

$\therefore$ Equation of tangent for $x^2=y$ is

$$y=mx-\frac{m^2}{4} \qquad (1)$$

$x^2=-y$, equation of tangent $y=mx+\frac{m^2}{4}$

$$y=m(x-2)+\frac{m^2}{4} \Rightarrow y=mx+\frac{m^2}{4}-2m \qquad (2)$$

from (1) and (2), we get

$$\frac{m^2}{4}-2m=\frac{-m^2}{4}$$

$$\frac{m^2}{2}=2m \Rightarrow m=0 \text{ or } 4$$

$\therefore$ Equation of tangent $y=4x-4$ or $y=0$

Option (b) (c) is correct

10.Sol: $y=-\frac{x^2}{2}+x+1$

$$2y=-x^2+2x+2$$

$$2y=-\left(x^2-2x\right)+2$$

$$2y=-\left[\left(x-1\right)^2-1\right]+2$$

$$2y=-\left(x-1\right)^2+1+2$$

$$\left(2y-3\right)=-\left(x-1\right)^2 \text{ or } \left(x-1\right)^2=-2\left(y-\frac{3}{2}\right)$$

Axis of parabola $\Rightarrow (x=1)$

It will be symmetric about $(x=1)$

Statement 1 is true

Statement 2 is true and explains Statement 1

Option a is correct

11.Sol: $y^2=4x$ and $x^2+y^2-6x+1=0$

solving the two equations

$$x^2+4x-6x+1=0$$

$$x^2-2x+1=0$$

$$(x-1)^2=0 \Rightarrow x=1$$

When $x=1$, $y=\pm 2$ for $y^2=4x$

When $x=1$, $y^2=\pm 2$ for $x^2+y^2-6x+1=0$

Points of meeting are $A(1, 2)$ and B (1, -2).

Slope of tangents at point A (for parabola) is

$$2yy^1=4, \quad y^1=\frac{2}{y}=1$$

Slope of tangents at A for circle

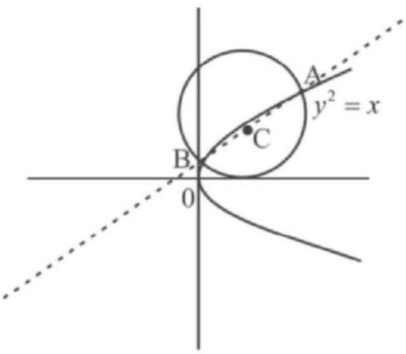
$$2x+2yy'-6=0$$

$$2yy' = 4$$

$$y' = 1$$

At point A the curves touches each other.
At point B
Slope of tangent for parabola is -1
Slope of tangent for circle is -1
At point B the curves touches each other
Option b is correct

12.Sol:



$A(t_1^2, 2t_1)$, $B(t_2^2, 2t_2)$, $C = \left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$

radius $= r = t_1 + t_2$.

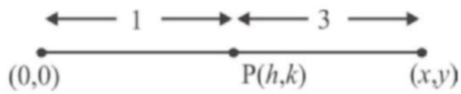
Slope of $AB = \frac{2t_1 - 2t_2}{t_1^2 - t_2^2}$

$$= \frac{2}{t_1 + t_2} = \frac{2}{r}.$$

Option c is correct.

13.Sol: $h = \frac{x+0}{4}$

$$k = \frac{y+0}{4}$$



$\Rightarrow \quad y^2 = 4x$

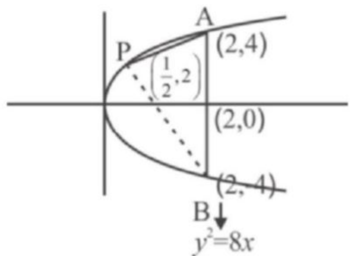
i.e., $\quad (4k)^2 = 4(4h)$

$\Rightarrow \quad k^2 = h$

$\therefore \quad y^2 = x$ is the locus of P

Option c is correct

14.Sol:



We know that the area of the triangle inscribed in a parabola is twice the area of the triangle formed by the tangents at the vertices of the triangle.

So tangents at $P\left(\frac{1}{2}, 2\right)$ and end points of latus rectum from triangle (Δ_1) is a half of area of triangle formed by the points P and end points of latus rectum (Δ_2). So $\frac{\Delta_1}{\Delta_2} = 2$

JEE-ADVANCED EVA-AITS - 3

"A Colossal juncture to get introduced to the national standard mock tests of JEE ADVANCED"

Section I (maximum Marks :32)

- This section has **EIGHT**questions .
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9 , both inclusive .
- For each question , type the correct integer in the space provided below the question using provided number keys.
- Marking scheme:
  - +4 if the correct integer is typed in the provided space
  - 0 in all other cases

1. If the area bounded by the curve $y^2 = x - y, y = -1$ and tangents to curve at the origin is Δ then $\frac{1}{\Delta}$ is

2. If the curve satisfying $x\,dx = \left(\frac{x^2}{y} - y^3\right)dy$ passes through $(0,-2)$, then value of $\{y(4)\}^2 . \left\{4 - \{y(4)\}^2\right\}$ is M^2, then $|M|$ is

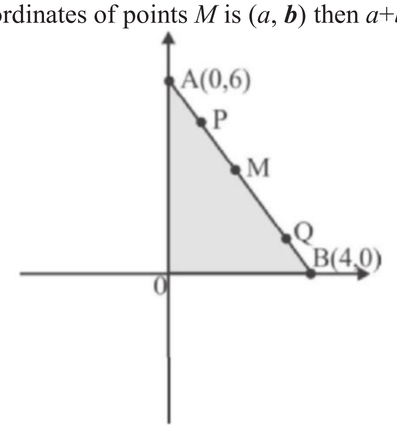
3. If *f(x)* is differentiable function and $\int_o^{\sin(x)} x\, f(x)dx = \sin(x)$, then $\left\{f\left(\frac{1}{2}\right)\right\}^2$ equals

4. Let m and m^2 are slopes of lines represented by the line pair $qx^2 - 2pxy + y^2 - 2x + 3y + \lambda = 0$ then value of $\frac{q + q^2 + 6pq}{p^3}$ is

5. If $a + b + c = 8, ab + ba + ca = 12$ $(\text{where } a, b, c \in R)$ then number of possible integral values of a is

6. If α, β, γ are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at which normals are concurrent, then $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)$ is equal to

7. Figure shows five collinear points A, P, M, Q and B such that $2PM = PQ$ and $AQ = BP$. If the coordinates of points M is (a, **b**) then a+b is



8. Let $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lambda k^4 + 2k^3 + k^2 + k + 1}{3n^5 + n^2 + n + 5k} = \frac{1}{3}$ then λ is equal to ...

Section 2 (Maximum Marks :32)

- This section has **EIGHT** questions.
- Each question has **FOUR**options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct
- For each question, select all the correct option(s) provided below the questions
- Marking scheme:
  - +4 if the correct integer is typed in the provided space
  - 0 if none of the options are selected
  - -2 in all other cases

1. If $g(x)$ is a continuous function such that $\int_0^x g(t)dt \to \infty,$ as $|x| \to \infty$, Then the value of k for which line $y = kx$ intersect the curve $\int_0^x g(t)dt = 2 - y^2$

(a) -1 (b) $\sqrt{2}$ (c) 3 (d) 1

2. If largest and smallest value of $\frac{y-4}{x-3}$ is p and q where (x, y) satisfy $x^2 + y^2 - 2x - 6y + 9 = 0$ then which of the following is true

(a) $p + q = \frac{4}{3}$ (b) q=1

(c) $p = \frac{4}{3}$ (d) $pq = \frac{4}{3}$

3. The most general solution of the differential equation $\dfrac{x + y\frac{dt}{dx}}{x - y\frac{dx}{dy}} = \frac{2y^3}{x^5}\sin^2\left(x^2 + y^2\right)$ is

(a) $-\frac{1}{2}\cot\left(x^2 + y^2\right) - \frac{2(y/x)^4}{4} + c = 0$

(b) $-\frac{1}{2}\cot\left(x^2 + y^2\right) - \frac{2(y/x)^4}{4} + e^c = 0$

(c) $-\frac{1}{2}\cot\left(x^2 + y^2\right) - \frac{2(y/x)^4}{4} + \tan c = 0$

(d) $\frac{1}{4}\tan\left(x^2 + y^4\right) - \frac{2y^3}{x} + c = 0$

4. x_1, x_2, x_3 are three real numbers satisfying the system of equations $x_1 + 3x_2 + 9x_3 = 27,\ x_1 + 5x_2 + 25x_3 = 125$ and $x_1 + 7x_2 + 49x_3 = 343$, then which of the following options are correct

(a) Number of divisors of $x_1 + x_3$ is 16

(b) $\frac{x_1 + x_2}{2}$ is a prime number

(c) $x_3 - x_2$ is a prime number

(d) $x_1 + x_2 + x_3$ is square of an integer

5. In a bag there are 10 black & 10 white balls. A ball is drawn at random & 5 extra balls of same color as of drawn ball are added in the bag along with the drawn ball. Now another ball is drawn and replaced in the bag but 4 balls of color same as drawn ball are removed from the bag. Again a ball is drawn and found to be white find the probability that the second drawn ball was black.

(a) $\frac{4}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) $\frac{1}{7}$

6. The normal to a curve at $P(x_1, y_1)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a

(a) Circle (b) Hyperbola
(c) Ellipse (d) Parabola

7. Let $a > 0, b > 0, c > 0$ and $a + b + c = 6$ then $\frac{(ab+1)^2}{b^2} + \frac{(bc+1)^2}{c^2} + \frac{(ca+1)^2}{a^2}$ may be

(a) $\frac{75}{4}$ (b) 35 (c) 15 (d) 10

8. If $g(x) = x^2 - x + 1$ and $f(x) == \sqrt{\frac{1}{x} - x}$, then

(a) Domain of $f(g(x))$ is [0, 1]

(b) Range of $f(g(x))$ is $\left(0, \frac{7}{2\sqrt{3}}\right]$

(c) $f(g(x))$ is many-one function

(d) $f(g(x))$ is unbounded function

Section III (Maximum Marks :16)

- This section contains **TWO** paragraphs.
- Based on each paragraph there will be **TWO** questions
- Each question has **FOUR** options . **ONE OR MORE THAN ONE** of these four option(s) is (are) correct
- For each question, select all the correct option(s) provided below the questions
- Marking scheme:
  - +4 if only all the correct option(s) is (are) selected
  - 0 if none of the option(s) are selected
  - -2 in all other cases

Paragraph - I

Consider a function

$f(x) = x^2 - ax^2 + (1-2a^2)x + a$, a is real.

1. The range of values of 'a' for which $f(x)$ has exact one real root

(a) $(0,\infty)$ (b) $(-\infty,1)$

(c) (-1,1) (d) $(1,\infty)$

2. If $a>1$ and α,β,γ are roots of $f(x)=0$ such that $\alpha<\beta<\gamma$,

(a) $\alpha<0,\beta<0,\gamma>1$

(b) $\alpha<-1,\beta>0,\gamma>0$

(c) $\alpha<0,-1<\beta<0,\gamma>0$

(d) $\alpha<0,\beta<0,\gamma<0$

Paragraph - II

A variable line L intersects the parabola $y=x^2$ at points P and Q whose x-coordinate are α and β respectively with $\alpha<\beta$ the area of the figure enclosed by the segment PQ and the parabola is always equal to $\frac{4}{3}$. The variable segment PQ has its midpoint M.

3. Which of the following is/are correct?

(a) $(\beta-\alpha)$ can have more than one real values

(b) $(\beta-\alpha)$ can be equal to 2

(c) $(\beta-\alpha)$ can have exactly one real value

(d) $\alpha = 2+\beta$

4. Which of the following is/are correct ?

(a) Equation of the pair of tangents, drawn to the curve, represented by locus of M from origin are $y=2x$ and $y=-2x$

(b) Equation of pair of tangents to the curve, represented by locus of M from origin are $y=x$ and $y=-x$.

(c) Area of the region enclosed between the curve represented by locus of M, and the pair of tangents drawn to it from origin is $\frac{2}{3}$ sq.units.

(d) Area of the region enclosed between the curve, represented by locus of M, and the pair of tangents drawn to it , from origin is $\frac{1}{3}$ sq. units.

BITSAT EVA-AITS - 1

"A Colossal juncture to get introduced to the national standard mock tests of BITSAT"

1. The centre of circle inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$
(a) (4,9) (b) (9,4) (c) (7,4) (d) (4,7)

2. A point moves such that the sum of its distance from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always 2a. Then equation of its locus is
(a) $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ (b) $\frac{x^2}{a^2} - \frac{y^2}{a^2(1-e^2)} = 1$
(c) $\frac{x^2}{a^2(1-e^2)} + \frac{y^2}{a^2} = 1$ (d) None of these

3. If $|x| < 1$ then the coefficient of x^n in the expansion of $(1 + x + x^2 + ...)^2$ will be
(a) 1 (b) n
(c) n+1 (d) None of these

4. The number of reflexive relations of a set with four elements is equal to
(a) 2^{16} (b) 2^{12} (c) 2^8 (d) 2^4

5. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is
(a) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$
(b) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
(c) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
(d) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

6. If $\sin x + \cos x = \frac{1}{5}$, then $\tan 2x$ is
(a) $\frac{25}{17}$ (b) $\frac{7}{25}$ (c) $\frac{25}{7}$ (d) $\frac{24}{7}$

7. General solution of $\sin x + \cos x = \min\limits_{a \in IR}\{1, a^2 - 4a + 6\}$ is
(a) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{4}$ (b) $2n\pi + (-1)^n \frac{\pi}{4}$
(c) $n\pi + (-1)^{n+1} \frac{\pi}{4}$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$

8. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by the second plane is
(a) 0.06 (b) 0.14 (c) 0.2 (d) 0.7

9. The minors of -4 and 9 and the co-factors of -4 and 9 in determinant $\begin{vmatrix} -1 & -2 & 3 \\ -4 & -5 & -6 \\ -7 & 8 & 9 \end{vmatrix}$ are respectively.

(a) 42, 3; -42, 3 (b) -42, -3; 42, -3
(c) 42, 3; -42, -3 (d) 42, 3;42, 3

10. If $x = a(t - \sin t)$ and $y = a(1 - \cos t)$, then $\frac{dy}{dx} =$

(a) $\tan\left(\frac{t}{2}\right)$ (b) $-\tan\left(\frac{t}{2}\right)$

(c) $\cot\left(\frac{t}{2}\right)$ (d) $-\cot\left(\frac{t}{2}\right)$

11. The number of ways four boys can be seated around a round-table in four chairs of different colours is
(a) 24 (b) 12 (c) 23 (d) 64

12. Find the equation of the auxiliary circle of $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(a) $x^2 + y^2 = 7$ (b) $x^2 + y^2 = 25$

(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 16$

13. The equation of the chord of the circle $x^2 + y^2 = a^2$ having (x_1, x_2) as its mid-point is

(a) $xy_1 + yx_1 = a^2$ (b) $x_1 + y_1 = a$

(c) $xx_1 + yy_1 = x_1^2 + y_1^2$ (d) $xx_1 + yy_1 = a^2$

14. The equation of pair of lines joining origin to the point of intersection of $x^2 + y^2 = 9$ and $x + y = 3$ is

(a) $(x + y)^2 = 9$ (b) $x^2 + (3 - x)^2 = 9$

(c) $xy = 0$ (d) $(3 - x)^2 + y^2 = 9$

15. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then the values of $C_0 + 2C_1 + 3C_2 + ...(n+1)C_n$ will be

(a) $(n+2)2^{n-1}$ (b) $(n+1)2^n$

(c) $(n+1)2^{n-1}$ (d) $(n+2)2^n$

16. The function $f(x) = \log(x + \sqrt{x^2 + 1})$, is
(a) An even function
(b) An odd function
(c) A periodic function
(d) Neither an even nor odd function

17. If $A + B + c = 180°$, then the value of $(\cot B + \cot C)\ (\cot C + \cot A)(\cot A + \cot B)$ will be
(a) $\sec A \sec B \sec C$
(b) $\cos ecA \cos ecB \cos ecC$
(c) $\tan A \tan B \tan C$
(d) 1

18. The sides of triangle are $3x + 4y, 4x + 3y$ and $5x + 5y$ units,where $x, y > 0$. The triangle is
(a) Right angled (b) Equilateral
(c) Obtuse angled (d) None of these

19. 8 coins are tossed simultaneously. The probability of getting atleast 6 heads is

(a) $\frac{57}{64}$ (b) $\frac{229}{256}$ (c) $\frac{7}{64}$ (d) $\frac{37}{256}$

20. A particle moves in a straight line so that it covered a distance $at^3 + bt + 5$ metre in t seconds. If its acceleration after 4 seconds is 48 metre/$(\text{sec})^2$, then a is equal to
(a) 1 (b) 2 (c) 3 (d) 4

21. An edge of a variable cube is increasing at the rate of 10 cm/sec. How fast the volume of the cube will increase when the edge is 5 cm long?

(a) $750cm^3$ / sec (b) $75cm^3$ / sec

(c) $300cm^3$ / sec (d) $150cm^3$ / sec

22. If ${}^nP_r = 840, {}^nC_r = 35,$ then n is equal to
(a) 1 (b) 3 (c) 5 (d) 7

23. The equation of a hyperbola whose asymptotes are $3x \pm 5y = 0$ and vertices are $(\pm 5, 0)$ is

(a) $3x^2 - 5y^2 = 25$ (b) $5x^2 - 3y^2 = 225$

(c) $25x^2 - 9y^2 = 225$ (d) $9x^2 - 25y^2 = 225$

24. The equation of the circle passing through the points of intersection of $x^2 + y^2 - 1 = 0,$ $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is

(a) $x^2 + y^2 + x + 2y = 0$

(b) $x^2 + y^2 - x + 20 = 0$

(c) $x^2 + y^2 - x - 2y = 0$

(d) $2(x^2 + y^2) - x - 2y = 0$

25. If $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & x\neq 0\\ 2, & x=0\end{cases}$ then

(a) $\lim\limits_{x\to 0+} f(x)\neq 2$

(b) $\lim\limits_{x\to 0-} f(x)=0$

(c) $f(x)$ is continuous at $x=0$

(d) None of these

26. Rank of matrix $\begin{bmatrix}4&1&0&0\\3&0&1&0\\6&0&2&0\end{bmatrix}$ is

(a) 4 (b) 3 (c) 2 (d) 1

27. The equation of tangent at (-4, -4) on the curve $x^2=-4y$ is

(a) $2x+y+4=0$ (b) $2x-y-12=0$

(c) $2x+y-4=0$ (d) $2x-y+4=0$

28. The equation of the normal to the curve $y^4=ax^3$ at (a,a) is

(a) $x+2y=3a$ (b) $3x-4y+a=0$

(c) $4x+3y=7a$ (d) $4x-3y=0$

29. The equation $(m-n)x^2+(n-l)x+l-m=0$ has equal roots, then l, m and n satisfy

(a) $2l=m+n$ (b) $2m=n+l$

(c) $m=n+l$ (d) $l=m+n$

30. $\dfrac{1+\dfrac{2^2}{2!}+\dfrac{2^4}{3!}+\dfrac{2^6}{4!}+....\infty}{1+\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{2^2}{4!}+...\infty}=$

(a) e^2 (b) e^2-1

(c) $e^{3/2}$ (d) None of these

31. The values of 'a' for which $(a^2-1)x^2+2(a-1)x+2$ is positive for any x are

(a) $a\geq 1$ (b) $a\leq 1$

(c) $a>-3$ (d) $a<-3\, or\, a>1$

32. $\displaystyle\int\frac{dx}{1+3\sin^2 x}=$

(a) $\frac{1}{3}\tan^{-1}(3\tan^2 x)+c$ (b) $\frac{1}{2}\tan^{-1}(2\tan x)+c$

(c) $\tan^{-1}(\tan x)+c$ (d) None of these

33. Everybody in a room shakes hand with everybody else. The total number of hand shakes is 66. The total number of persons in the room is

(a) 11 (b) 12 (c) 13 (d) 14

34. The value of k so that $x^2+y^2+kx+4y+2=0$ and $2(x^2+y^2)-4x-3y+k=0$ cuts orthogonally is

(a) $\frac{10}{3}$ (b) $\frac{-8}{3}$ (c) $\frac{-10}{3}$ (d) $\frac{8}{3}$

35. The function $f(x)=\begin{cases}\dfrac{\tan x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$ is

(a) Continuous but not differentiable at $x=0$

(b) Discontinuous at $x=0$

(c) Continuous and differentiable at $x=0$

(d) Not defined at $x=0$

36. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k}{n^2+k^2}$ is equals to

(a) $\frac{1}{2}\log 2$ (b) $\log 2$

(c) $\pi/4$ (d) $\pi/2$

37. The 5^{th} term of the series $\frac{10}{9},\frac{1}{3}\sqrt{\frac{20}{3}},\frac{2}{3}....$ is

(a) $\frac{1}{3}$ (b) 1 (c) $\frac{2}{5}$ (d) $\sqrt{\frac{2}{3}}$

38. $\cot^{-1}[(\cos\alpha)^{1/2}]-\tan^{-1}[(\cos\alpha)^{1/2}]=x$, then $\sin x=$

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$

(c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

39. $\lim_{x\to 0}\frac{\sin mx}{\tan nx}=$

(a) $\frac{n}{m}$ (b) $\frac{m}{n}$

(c) mn (d) None of these

40. If the mean of 3, 4, x, 7, 10 is 6, then the value of x is

(a) 4 (b) 5 (c) 6 (d) 7

41. If $x_r=\cos\left(\frac{\pi}{2^r}\right)+i\sin\left(\frac{\pi}{2^r}\right)$,

then $x_1, x_2 \ldots \infty$ is

(a) -3 (b) -2 (c) -1 (d) 0

42. The sum of the first five terms of the series

$3+4\frac{1}{2}+6\frac{3}{4}+\ldots$ will be

(a) $39\frac{9}{16}$ (b) $18\frac{3}{16}$

(c) $39\frac{7}{16}$ (d) $13\frac{9}{16}$

43. Solution of the differential equation

$\cos x\,dy = y(\sin x - y)dx, 0<x<\frac{\pi}{2}$ is

(a) $\sec x=(\tan x+c)y$ (b) $y\sec x=\tan x+c$

(c) $y\tan x=\sec x+c$ (d) $\tan x=(\sec x+c)y$

44. If H is the harmonic mean between p and q, then the value of $\frac{H}{p}+\frac{H}{q}$ is

(a) 2 (b) $\frac{pq}{p+q}$

(c) $\frac{p+q}{pq}$ (d) None of these

45. Sum of n terms of the following series

$1^3+3^3+5^3+7^3+\ldots$ is

(a) $n^2(2n^2-1)$ (b) $n^3(n-1)$

(c) n^3+8n+4 (d) $2n^4+3n^2$

PROBLEM OF THE MONTH

EXCLUSIVE AREAS

1. A and B lie on the circumference of the circle with centre O, radius 2, and $\angle AOB = 90^0$. Another circle, with diameter AB is drawn. O lies on the circumference of this second circle. The unshaded region is the area common to both circles. The shaded region is the area in one circle or the other circle but in both. Determine the area of the shaded region in the diagram.

Solution to the above problem will be published in the next issue.

CONCEPT EXPLORATION

SET AND ELEMENTS RELATIONS

A relation on a set A is just a set of pairs of elements of A, (possible the same). Note that we consider ordered pairs - (x,y) is not the same as (y,x).
An important point is that you can define relations quite arbitrarily. For every possible pair (x,y), you can decide whether or not you include it in the relation - a relation has nothing to do with properties or " logical relationships". A relation is just a set.
In the case of a finite set A (say of n elements), there is a simple interpretation of a relation. We simply drawan n table, representing all the possible pairs (x,y), and we put a '*' in a cell when the corresponding pair belongs to the relation. For example, with the set $A = a, b, c$ we could have the following relation:

	a	b	c
a		*	
b			
c	*		*

In this case, the relation contains the pairs (a,b), (c,a), and (c,c).
In general, for every way you can put stars in the above table (including none at all), you get a relation on A.
We will first examine a few simpler problems.

(I) All relations

As a first exercise, let us count how many relations are possible on this set.
We have $3 \times 3 = 9$ squares to fill. For each square, we can decide either to include it (put a '*' in it), or not. This makes two possibilities for each square. When you combine the nine squares, you have a total of $2^9 = 512$ possibilities - there are 512 possible relations on A. In general, for a set of n elements, there are n^2 squares in the table, and $2^{(n^2)}$ possible relations.

(II) Reflexive relations

A relation is reflexive if it contains all the pairs (x,x) for every x in A. In the example above, this means that we must have '*' in all squares of the main diagonal - the smallest possible reflexive relation is:

	a	b	c
a	*		
b		*	
c			*

In a reflexive relation, the three squares of the diagonal are fixed. You are still free to include or not any of the 6 remaining squares - this gives a total of $2^6 = 64$ possibilities.
For a set of n elements, you would have:

$$2^{(n^2 - n)} = 2^{(n(n-1))}$$

possible reflexive relations.

(III) Irreflexive relations

A relation is irreflexive if it contains none of the pairs (x,x). This means that you must have no '*' on the main diagonal, and you are still free to do whatever you want with the other squares. The number of irreflexive relations is therefore the same as the number of reflexive relations.

Note that " irreflexive" is not the same as "not reflexive". "Irreflexive" means you have no squares on the diagonal; "not reflexive" means you don't have all the squares on the diagonal. The very first example of this message is neither reflexive (since it does not contain (a,a)) nor irreflexive (since it contains (c,c)).

(IV) Symmetric relations

A relation is symmetric if, whenever it contains the pair (x,y), it also contains the pair (y,x).

This means that the table must be symmetric with respect to the main diagonal. For example, the following is a symmetric relation:

	a	b	c
a		*	
b	*		*
c		*	*

We note that the off-digonal elements come in pairs: (a,b) and $(b,a),(b,c)$ and (c,b). The diagonal elements are not taken into account.

To build a symmetric relation, we can freely choose all the squares on and above the diagonal.

There are $\frac{n(n+1)}{2}$ such squares, and two possibilities for each of them, so the number of symmetric relation is:

$$2^{\frac{n(n+1)}{2}}$$

(V) Antisymmetric relations

A relation is antisymmetric if, whenever it contains both (x,y) and $(y,x), x=y$ (x and y are the same element). This is equivalent to saying that, if x and y are distinct elements, you cannot have at the same time (x,y) and (y,x) in the relation.

The following is an example of an antisymmetric relation:

	a	b	c
a	*	*	
b			
c	*		*

Note that the relation contains (a,b) but not (b,a), and (c,a) but not (a,c). Also, it contains neither of (b,c) and (c,b). Elements on the diagonal can be selected freely.

How many antisymmetric relations are there?

We count separately the possibilities for diagonal and off-diagonal elements.

For diagonal elements, there are two possibilities for each of them, and there are n such elements.

This gives 2^n possibilities.

For each pair of off-diagonal elements x and y, we have three possibilities:

(x,y)	(y,x)
out	out
in	out
out	in

since we cannot have both (x,y) and (y,x) in the relation. The number of pairs of distinct elements is "n choose 2":

$$\binom{n}{2}=\frac{n(n-1)}{2}$$

and, as there are three possibilities for each pair, we have $3^{\frac{n(n-1)}{2}}$ possibilities for off-diagonal elements.

The total number of antisymmetric relations is thus:

$$2^n \cdot 3^{\frac{n(n-1)}{2}}$$

Note that "antisymmetric" is not the same as "not symmetric". For example, the following relation is neither symmetric nor antisymmetric:

	a	b	c
a		*	*
b			
c	*		*

It is not symmetric, because it contains (a,b) but not (b,a), and it is not antisymmetric because it contains both (a,c) and (c,a). Now, let us come to your specific questions they are, in fact, a little easier.

(VI) Reflexive and antisymmetric

If you compare that with the antisymmetric case, the only difference is that you must have '*' in all diagonal squares-you are no longer free to select them. You still have 3 possibilities for each of the $\frac{n(n-1)}{2}$ pairs of distinct elements (off-diagonal squares), and the total number is therefore:

$$3^{\frac{n(n-1)}{2}}$$

JEE-ADVANCED EVA-AITS - 4

"A Colossal juncture to get introduced to the national standard mock tests of JEE ADVANCED"

Section 1 (Maximum Marks :18)

- This section has **SIX** questions
- Each question has **FOUR** options (a),(b),(c), and (d). ONLY One of these four options is correct
- For each question, select the alphabets corresponding to all the correct options provided below the questions
- Marking Scheme:
  - FULL MARKS: +3 if only the alphabet corresponding to the correct option is selected
  - ZERO MARKS: 0 if none of the alphabet is selected
  - NEGATIVE MARKS:-1 In all others cases

1. A is a matrix of order 3×3 and a_{ij} is its elements of i^{th} row and j^{th} column. If $a_{ij}+a_{jk}+a_{ki}=0$ holds for all $1\le i,j,k\le 3$ then

(a) A is a non-singular matrix

(b) A is a singular matrix

(c) $\sum\limits_{1\le i,j\le 3} a_{ij}$ is equal zero

(d) A is a symmetric matrix

2. f and g are two real valued continuous functions and let $\int g(x)\,dx=f^{-1}(x)$ and $f(x)=x^3+x+\sin \pi x+2$ then the value of $\int\limits_2^4 xg(x)\,dx$ is

(a) $\frac{-11}{4}-\frac{2}{\pi}$ (b) $\frac{11}{4}-\frac{2}{\pi}$

(c) $\frac{-11}{4}+\frac{2}{\pi}$ (d) $\frac{11}{4}+\frac{2}{\pi}$

3. Two circle with radii r_1 and r_2 respectively touch each other externally. Let r_3 be the radius of a circle that touches these two circles as well as a common tangents to two circles then which of the following relation is true

(a) $\frac{1}{\sqrt{r_3}}=\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_2}}$ (b) $\frac{1}{\sqrt{r_3}}=\left|\frac{1}{\sqrt{r_1}}-\frac{1}{\sqrt{r_2}}\right|$

(c) $\sqrt{r_3}=\sqrt{r_1}+\sqrt{r_2}$ (d) $\sqrt{r_3}=\left|\sqrt{r_1}-\sqrt{r_2}\right|$

4. If $S_n=3+\frac{1+3+3^2}{3!}+\frac{1+3+3^2+3^3}{4!}$..... upto n-terms. Then the value of $\left[\lim\limits_{n\to\infty} S_n\right]$ is, (where [.] represents G.I.F)

(a) 6 (b) 7 (c) 8 (d) 9

5. If

$$\int\limits_{-20}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx+\int\limits_{\frac{1}{21}}^{\frac{1}{11}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx+\int\limits_{\frac{21}{10}}^{\frac{11}{10}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2=l.$$

then $l+\frac{420}{7939}$ is equal to

(a) $\frac{110}{939}$ (b) $\frac{110}{969}$ (c) $\frac{110}{739}$ (d) $\frac{120}{759}$

6. If $P = \operatorname{cosec}\frac{\pi}{8} + \operatorname{cosec}\frac{2\pi}{8} + \operatorname{cosec}\frac{3\pi}{8}$ $+ \operatorname{cosec}\frac{13\pi}{8} + \operatorname{cosec}\frac{14\pi}{8} + \operatorname{cosec}\frac{15\pi}{8}$, and $Q = 8\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$, then value of $P+Q$ is

(a) 0 (b) 1 (c) 2 (d) 3

Section 2 (Maximum Marks:32)

- This section contains **EIGHT** questions
- Each question has **FOUR** options (a),(b),(c), and (d). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct
- For each question, select the alphabets corresponding to all the correct option(s) provided below the questions
- Marking scheme:
  - FULL MARKS: +4 if only the alphabets corresponding to all the correct option(s) is(are) selected option, provided NO incorrect option alphabet is selected
  - ZERO MARKS: 0 if none of the alphabets are selected
  - NEGATIVE MARKS: -2 in all other cases
- For Example: If (a),(c) and (d) are all correct options for a question, selecting alphabets corresponding to all these three options will result in +4 marks. Selecting only (a) and (d) will result in +2 marks; selecting (a) and (b) will result in -2 marks , as an alphabet corresponding to wrong option is also selected

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & 0 & k \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - \alpha x + \beta = 0$. If A satisfies $f(x) = 0$, then

(a) $k = 1, \alpha = 14$ (b) $\alpha = 14, \beta = 22$

(c) $k = -1, \beta = 22$ (d) $\alpha = -14, \beta = -22$

2. If the expression $kx^2 + (2k-1)xy + y^2 + 2x - 2ky$ can be resolved as a product of two linear factors, then

(a) There exists no real value of k

(b) Atleast one value of k is negative

(c) For atleast one real value of $k, 3k^3 + 1$ is negative

(d) There exists no real value of k for which $3k^3 + 1$ is negative

3. $ABCD$ is a regular tetrahedron. P & Q are the mid-points of the edges AC and AB respectively, G is the centroid of the face BCD and θ is the angle between the vectors $\overrightarrow{PG}$ and $\overrightarrow{DQ}$, then

(a) The angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is $90°$

(b) The angle θ is $\pi - \cos^{-1}\left(\frac{5}{3\sqrt{3}}\right)$

(c) The angle θ is $\pi - \cos^{-1}\left(\frac{5}{6\sqrt{3}}\right)$

(d) The angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is $120°$

4. All x in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$ is

(a) $\frac{\pi}{12}$ (b) $\frac{11\pi}{36}$ (c) $\frac{13\pi}{36}$ (d) $\frac{\pi}{6}$

5. Let $S_n = \sum_{r=1}^{n}\left(\frac{r^4 + r^3 n + r^2 n^2 + 2n^4}{n^5}\right)$ and

$$T_n = \sum_{r=0}^{n-1}\left(\frac{r^4 + r^3 n + r^2 n^2 + 2n^4}{n^5}\right), \quad n = 1, 2, 3, \ldots$$

then

(a) $T_n > \frac{167}{60}$ (b) $T_n < \frac{167}{60}$

(c) $S_n > \frac{167}{60}$ (d) $S_n < \frac{167}{60}$

6. If z_1, z_2, z_3, z_4 are complex numbers in an Argand plane satisfying $z_1 + z_3 = z_2 + z_4$. A complex

number 'z' lies on the line joining z_1 and z_4 such that $\operatorname{Arg}\left(\frac{z-z_2}{z_1-z_2}\right)=\operatorname{Arg}\left(\frac{z_3-z_2}{z-z_2}\right)$. It is given that $|z-z_4|=5, |z-z_2|=|z-z_3|=6$ then

(a) Area of the triangle formed by z, z_1, z_2 is $3\sqrt{7}$ sq.units

(b) Area of the triangle formed by z, z_3, z_4 is $\frac{15\sqrt{7}}{4}$ sq. units

(c) Area of the quadrilateral formed by the points z_1, z_2, z_3, z_4 taken in order is $\frac{27\sqrt{7}}{2}$ sq. units

(d) Area of the quadrilateral formed by the points z_1, z_2, z_3, z_4 taken in order is $\frac{27\sqrt{7}}{4}$ sq. units.

7. The vertices of a triangle ABC are $A(2,0,2)$, $B(-1,1,1)$ and $C(1,-2,4)$. The points D and E divided the sides AB and CA in the ratio 1:2 respectively. Another point F is taken in space such that the perpendicular drawn from F to the plane containing ΔABC, meets the plane at the point of intersection of the line segments CD and BE. If the distance of F from the plane of triangle ABC is $\sqrt{2}$ units, then

(a) The volume of the tetrahedron $ABCF$ is $\frac{7}{3}$ cubic units

(b) The volume of the tetrahedron ABCF is $\frac{7}{6}$ cubic units

(c) One of the equation of the line AF is $\vec{r}=(2\hat{i}+2\hat{k})+\lambda(2\hat{k}-\hat{i})\,(\lambda\in R)$

(d) One of the equation of the line AF is $\vec{r}=(2\hat{i}+2\hat{k})+\mu(\hat{i}+7\hat{k})$

8. If a,b,c are in A.P and A,B,C are in G.P (common ratio $\neq 1$). Then which of the following is/are correct.

(a) $\frac{A}{a},\frac{B}{b},\frac{C}{c}$ are in H.P if common ratio of G.P is $\frac{C}{a}$.

(b) $\frac{a}{A},\frac{b}{B},\frac{c}{C}$ are in H.P if common ratio of G.P is equal to common difference of A.P

(c) $\frac{A^2}{a},\frac{B^2}{b},\frac{C^2}{c}$ are in H.P if common ratio of G.P is $\sqrt{\frac{c}{a}}$

(d) $\frac{a}{A^2},\frac{b}{B^2},\frac{c}{C^2}$ are in H.P if common ratio of G.P is equal to square root of common difference of A.P.

Section 3(Maximum Marks :12)

- This section contains **TWO** paragraphs .
- Based on each paragraph there will be **TWO** questions
- Each question has **FOUR** options (a),(b),(c) and (d). **ONLY ONE** of these four options is correct
- For each question, select the alphabet corresponding to all the correct option provided below the questions.
- Marking scheme:
  - FULL MARKS: + 3 if only the alphabets corresponding to all correct option(s) is(are) selected.
  - ZERO MARKS: 0 In all other cases

Paragraph: 1

Let $X_i \in R, i=\{1,2,3...n\}$ are numbers such that

$$\sum_{i=1}^{n} i\sqrt{X_i - i^2} = \frac{\sum_{i=1}^{n} X_i}{2} \text{ and } X_1+X_2+...+X_n=280.$$

1. No, of ways of distribution of n identical objects among 3 persons such that each get atleast one object is
(a) 4 (b) 10 (c) 20 (d) 140

2. Probability that a randomly selected triangle formed by vertices of a $2n+1$ sided regular polygon is isosceles is
(a) $\frac{3}{13}$ (b) $\frac{5}{13}$ (c) $\frac{7}{13}$ (d) $\frac{9}{13}$

Paragraph: 2

Column-1: Real valued function ;
Column-2: Continuity of the function ;
Column-3: Differentiability of the function

Column -1	Column -2	Column -3
I. $f(x)=\lvert\lvert x-6\rvert-\lvert x-8\rvert\rvert-\lvert x^2-4\rvert+3x-\lvert x-7\rvert^3$	(i) Continuous, $\forall x\in R$	(P) Not differentiable at 3 points
II. $f(x)=(x^2-9)\lvert x^2+11x+24\rvert+\sin\lvert x-7\rvert$ $+\cos\lvert x-4\rvert+(x-1)^{3/5}\sin(x-1)$	(ii) Discontinuous at a single point only	(Q) Not differentiable at 4 points
III. $f(x)=\begin{cases}(x+1)^{3/5}-\frac{3\pi}{2} & ;\ x<-1\\ \left(x-\frac{1}{2}\right)\cos^{-1}(4x^3-3x) & ;-1\le x\le 1\\ (x-1)^{5/3} & ;1<x<2\end{cases}$	(iii) Discontinuous at 2 points	(R) Not differentiable at 2 points
IV. $f(x)=\{\sin x\}\{\cos x\}+\{\sin^3\pi x\ \{x\}\}([x])\ x\in[-1,2\pi]$	(iv) Discontinuous at 3 points	(S) Not differentiable at 5 points

Match the following columns (s)

3. Which of the following combination is correct
(a) (I) (i) (R) (b) (III) (ii) (R) (c) (IV) (iv) (P) (d)(I) (i) (Q)

4. Which of the following combination is correct
(a) (II) (iii) (S) (b) (III) (i) (P) (c) (II) (iii) (R) (d) (III) (i) (R)

BIT SAT EVA-AITS - 2

"A Colossal juncture to get introduced to the national standard mock tests of BITSAT"

1. The equation of the circle which passes through the points of intersection of the circles $x^2+y^2-6x=0$ and $x^2+y^2-6y=0$ and has its centre at $\left(\frac{3}{2},\frac{3}{2}\right)$ is

(a) $x^2+y^2+3x+3y+9=0$

(b) $x^2+y^2+3x+3y=0$

(c) $x^2+y^2-3x-3y=0$

(d) $x^2+y^2-3x-3y+9=0$

2. The angle between the tangents from (α,β) to the circle $x^2+y^2=a^2$, is

(a) $\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$

(b) $\tan^{-1}\left(\frac{\sqrt{\alpha^2+\beta^2-a^2}}{a}\right)$

(c) $2\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$

(d) None of these

3. The locus of mid point of the chords of the circle $x^2+y^2-2x-2y-2=0$ which makes an angle of 120^0 at the centre is

(a) $x^2+y^2-2x-2y+1=0$

(b) $x^2+y^2+x+y-1=0$

(c) $x^2+y^2-2x-2y-1=0$

(d) None of these

4. A chord AB drawn from the point A (0,3) on circle $x^2+4x+(y-3)^2=0$ meets to M in such a way that $AM=2AB$, then the locus of point M will be

(a) Straight line

(b) Circle

(c) Parabola

(d) None of these

5. The equation of the line joining the point (3,5) to the point of intersection of the lines $4x+y-1=0$ and $7x-3y-35=0$ is equidistant from the points (0,0) and (8,34)

(a) True

(b) False

(c) Nothing can be said

(d) None of these

6. If one of the lines of the pair $ax^2+2hxy+by^2=0$ bisects the angle between positive directions of the axes, then a,b,h satisfy the relation

(a) $a+b=2|h|$

(b) $a+b=-2h$

(c) $a-b=2|h|$

(d) $(a-b)^2=4h^2$

7. In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as

(a) $f(0)=1/e$ (b) $f(0)=0$

(c) $f(0)=e$ (d) None of these

8. Which of the following is not true

(a) Every differentiable function is continuous

(b) If derivative of a function is zero at all points, then the function is constant

(c) If a function has maximum or minimum at a point, then the function is differentiable at that point and its derivative is zero

(d) If a function is constant, then its derivative is zero at all points

9. Let α,β be the roots of $x^2-2x\cos\phi+1=0$, then the equation whose roots are α^n,β^n is

(a) $x^2-2x\cos n\phi-1=0$

(b) $x^2-2x\cos n\phi+1=0$

(c) $x^2-2x\sin n\phi+1=0$

(d) $x^2+2x\cos n\phi-1=0$

10. If the roots of the equation $x^2-2ax+a^2+a-3=0$ are real and less than 3, then

(a) $a<2$

(b) $2\le a\le 3$

(c) $3<a\le 4$

(d) $a>4$

11. If $C_0,C_1,C_2,....,C_n$ are the binomial coefficients, then $2.C_1+2^3.C_3+2^5.C_5+....$ equals

(a) $\frac{3^n+(-1)^n}{2}$ (b) $\frac{3^n-(-1)^n}{2}$

(c) $\frac{3^n+1}{2}$ (d) $\frac{3^n-1}{2}$

12. The sum of the coefficients in the expansion of $(1+x-3x^2)^{2163}$ will be

(a) 0 (b) 1

(c) -1 (d) 2^{2163}

13. For any two sets A and B if $A\cap X=B\cap X=\phi$ and $A\cup X=B\cup X$ for some set X, then

(a) $A-B=A\cap B$

(b) $A=B$

(c) $B-A=A\cap B$

(d) None of these

14. If $f(x)=\sin^2 x$ and the composite function $g\{f(x)\}=|\sin x|$, then the function $g(x)$ is equal to

(a) $\sqrt{x-1}$ (b) $\sqrt{x}$

(c) $\sqrt{x+1}$ (d) $-\sqrt{x}$

15. In a triangle $PQR, \sqrt{R}=\frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2+bx+c=0\,(a\ne 0)$. Then

(a) $a+b=c$

(b) $b+c=a$

(c) $a+c=b$

(d) $b=c$

16. If $\frac{\cos A}{3}=\frac{\cos B}{4}=\frac{1}{5}, -\frac{\pi}{2}<A<0, -\frac{\pi}{2}<B<0$ then value of $2\sin A+4\sin B$ is

(a) 4 (b) -2

(c) -4 (d) 0

17. Let $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$ where $x\in R$ and $k\ge 1$. Then $f_4(x)-f_6(x)$ equals

(a) $\frac{1}{4}$ (b) $\frac{1}{12}$

(c) $\frac{1}{6}$ (d) $\frac{1}{3}$

18. $\cot\theta=\sin 2\theta(\theta\ne n\pi, n$ is integer), if $\theta=$

(a) 45^0 and 60^0 (b) 45^0 and 90^0

(c) 45^0 only (d) 90^0 only

19. Two straight roads intersect at an angle of 60^0. A bus on one road is 2 *km* away from the intersection and a car on the other road is 3 *km* away from the intersection. Then the direct distance between the

two vehicles is

(a) 4 km (b) $\sqrt{2}$ km

(c) 4 km (d) $\sqrt{7}$ km

20. The probability that a leap year will have 53 Fridays or 53 Saturdays is

(a) $\frac{2}{7}$ (b) $\frac{3}{7}$

(c) $\frac{4}{7}$ (d) $\frac{1}{7}$

21. An unbiased coin is tossed. If the result is a head, a pair of unbaised dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 4,...., 12 is picked and the number on the card is noted. The probability that the noted number is either 7 or 8, is

(a) 0.24 (b) 0.244

(c) 0.024 (d) None of these

22. If A, B, C be the angles of a triangle, then

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} =$$

(a) 1

(b) 0

(c) $\cos A \cos B \cos C$

(d) $\cos A + \cos B \cos C$

23. The value of λ for which the system of equations $2x - y - z = 12,\ x - 2y + z = -4,\ \ x + y + \lambda z = 4$ has no solution is

(a) 3 (b) -3 (c) 2 (d) -2

24. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ such that

(a) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $PX = X$

(c) $PX = 2X$ (d) $PX = -X$

25. If $y\sqrt{x^2+1} = \log\left\{\sqrt{x^2+1} - x\right\}$, then

$(x^2+1)\frac{dy}{dx} + xy + 1 =$

(a) 0 (b) 1

(c) 2 (d) None of these

26. For the function $f(x) = x^2 - 6x + 8, 2 \le x \le 4,$ the value of x for which $f'(x)$ vanishes, is

(a) $\frac{9}{4}$ (b) $\frac{5}{2}$ (c) 3 (d) $\frac{7}{2}$

27. The speed v of a particle moving along a straight line is given by $a + bv^2 = x^2$ (where x is its distance from the origin). The acceleration of the particle is

(a) bx (b) x/a (c) x/b (d) x/ab

28. The abscissa of the point on the curve $y = a\left(e^{x/a} + e^{-x/a}\right)$ where the tangent is parallel to the $x-$axis is

(a) 0 (b) a (c) $2a$ (d) $-2a$

29. In (-4,4) the function $f(x) = \int_{-10}^{x}(t^4 - 4)e^{-4t}dt$ has

(a) No extrema (b) One extremum

(b) Two extrema (d) Four extrema

30. $\int \sin^{-1}(3x - 4x^3)\,dx =$

(a) $x\sin^{-1}x + \sqrt{1-x^2} + c$

(b) $x\sin^{-1}x - \sqrt{1-x^2} + c$

(c) $2[x\sin^{-1}x + \sqrt{1-x^2}] + c$

(d) $3[x\sin^{-1}x + \sqrt{1-x^2}] + c$

31. Area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(a) $\frac{8}{9}$ $sq.unit$ (b) $\frac{9}{8}$ $sq.unit$

(c) $\frac{4}{3}$ $sq.unit$ (d) None of these

32. In how many ways can 6 persons be selected from 4 officers and 8 constables, if atleast one officer is to be included

(a) 224 (b) 672

(c) 896 (d) None of these

33. In a certain test there are n questions. In the test 2^{n-1} students gave wrong answers to atleast i questions, where $i = 1, 2, ...n$. If the total number of wrong answers given is 2047, then n is equal to

(a) 10 (b) 11 (c) 12 (d) 13

34. For $2 \le r \le n, \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$ is equal to

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$

(c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

35. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is

(a) $x = 1$ (b) $2x + 1 = 0$ (c) $x = -1$ (d) $2x - 1 = 0$

36. If 'a' and 'c' are the segments of a focal chord of a parabola and b the semi-latus rectum, then

(a) a, b, c are in A.P

(b) a, b, c are in G.P

(c) a, b, c are in H.P

(d) None of these

37. If $S_n = nP + \frac{1}{2}n(n-1)Q$, where S_n denotes the sum of the first n terms of an A.P., then the common difference is

(a) $P + Q$ (b) $2P + 3Q$

(c) $2Q$ (d) Q

38. The value of n for which $\frac{x^{n+1} + y^{n+1}}{x^n + y^n}$ is the geometric mean of x and y is

(a) $n = -\frac{1}{2}$ (b) $n = \frac{1}{2}$ (c) $n = 1$ (d) $n = -1$

39. The sum of the series $1 + 2x + 3x^2 + 4x^3 + ...$ upto n terms is

(a) $\frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}$ (b) $\frac{1 - x^n}{1 - x}$

(c) x^{n+1} (d) None of these

40. If $1 + \cos\alpha + \cos^2\alpha + ...\infty = 2 - \sqrt{2}$, then $\alpha, (0 < \alpha < \pi)$ is

(a) $\pi / 8$ (b) $\pi / 6$

(c) $\pi / 4$ (d) $3\pi / 4$

41. The trigonometric equation $\sin^{-1} x = 2\sin^{-1} 2a$ has a real solution if

(a) $|a| > \frac{1}{\sqrt{2}}$

(b) $\frac{1}{2\sqrt{2}} < |a| < \frac{1}{\sqrt{2}}$

(c) $|a| > \frac{1}{2\sqrt{2}}$

(d) $|a| \le \frac{1}{2\sqrt{2}}$

42. The value of $\lim_{x \to 0} \frac{2}{x} \log(1 + x)$ is equal to

(a) e (b) e^2 (c) 1/2 (d) 2

43. The equation of line of the curve whose slope at any point is equal to $y + 2x$ is

(a) $y = 2(e^x + x - 1)$

(b) $y = 2(e^x - x - 1)$

(c) $y = 2(e^x - x + 1)$

(d) $y = 2(e^x + x + 1)$

44. Suppose a population A has 100 observations 101, 102,..., 200 and another population B has 100 observations 151, 152, ..., 250. If V_A and V_B represent the variances of the two populations, respectively, then $\frac{V_A}{V_B}$ is

(a) 1 (b) $\frac{9}{4}$ (c) $\frac{4}{9}$ (d) $\frac{2}{3}$

45. If $|8 + z| + |z - 8| = 16$ where z is a complex number, then the point z will lie on

(a) A circle (b) An ellipse

(c) A straight line (d) None of these

VEDIC MATHEMATICS

FACTORISATION OF QUADRATICS

Multiplication of two factors to find their product is a straight forward process. Factorisation is the reverse process which starts with the product and discovers the original factors.

In general, factorisation is a more difficult process than multiplication. The public-key cryptographic system relies for its secrecy on the difficulty of factorising a large number of perhaps 40 or 50 digits which is the public key. Only those who know the factors can crack the code, though it would be a simple matter to calculate the key, given the factors.

Similarly, algebraic factorisation is more difficult than algebraic multiplication. The sutra Vertically and Crosswise deals with multiplication, and in the simplest case two linear factors give a quadratic product. In Vedic Mathematics Tirthaji also gives the sutras which are needed for factorisation. These are Anurupyena, or Proportionately and Adyamadyenantyamantyena, The first by the first and the last by the last.

Example 1: Consider the factorisation of the quadratic equation $x^2+7x+10.$ Knowing that the general expression $x^2+(a+b)x+ab$ has two linear factors $(x+a)$ and $(x+b)$, we wish to find two numbers a and b such that $a+b=7$ and $ab=10$. This is a very simple case and there is little difficulty in seeing that $a=2$ and $b=5$.

Example 2: Factorisation is more difficult when the coefficient of x^2 in the quadratic is not unity. Consider $3x^2+10x+8$. The factors must be of the form $(3x+a)(x+b)$, and we see that the independent terms a and b multiply to make 8. Maybe $a=1$ and $b=8$, or $a=2$ and $b=4$, but it becomes a matter of trial and error to see which of these combinations applies and to find which of a and b belongs to the $3x$ and which to the x.

This is where the sutras offer a systematic process in which the mind can come to rest. Like a jeweller shaping a gemstone, we are looking for the natural angle at which the stone will split. The crucial first step is to focus on the coefficient of the middle term. This should split into two parts such that the ratio of the coefficient of x^2 to the first part is the same the ratio of the second part to the independent term. For our example, $3x^2+10x+8,$ the middle coefficient 10 splits as $10=6+4,$ so that

$$3x^2+10x+8=3x^2+6x+4x+8$$

The ratio 3 : 6 is the same as the ratio 4 : 8. In its simplest terms the common ratio is 1: 2 and this shows that $(x+2)$ is one of the factors we are looking for. Once Proportionately has done its work, we use The first by the first and the last by the last to find the remaining factor. Dividing the first term x of the newly found factor $(x+2)$ into the first term $3x^2$ of the original quadratic we find the first term $3x$ of the second factor. And dividing the last term 2 of the $(x+2)$ into the last term 8 of the quadratic we get the last term 4 of the second factor.

So we say $3x^2+10x+8=(x+2)(3x+4)$

Another way to arrive at the same set of factors is via the split $10=6+4.$ This gives ratios 3 : 4

and 6 : 8, which though less obvious are perfectly valid. Working from this split we get the $(3x+4)$ factor first and then derive the $(x+2)$ factor via *The first by the first and the last by the last.*

We may notice that whichever route we follow we need pay no further attention to the middle term of the quadratic once the first linear factor is found.

Negative coefficients: The same principles apply when the quadratic contains negative coefficients but the procedure may be adapted to retain maximum simplicity, using the sutra *Paravartya Yojayet or Transpose and adjust.*

Example 3: Suppose the quadratic to be factorised is $2x^2+11x+15,$ where the term in x is negative but the independent term remains positive. Mentally, we first transpose the term $-11x$ to $+11x$, and consider the quadratic $2x^2+11x+15.$ Splitting the middle term coefficient as as $11=6+5$ gives ratios 2 : 6 and 5 : 15 and indicates a factor $(x+3)$. The second factor of the modified quadratic is then $(2x+5)$.

The last step is to adjust these two factors, putting minus signs in place of plus, to give the factors of the original quadratic. So finally

$$2x^2-11x+15=(x-3)(2x-5).$$

Example 4: Another situation which is just a little different from the point of view of mental working is when the independent term is negative, as with $3x^2+2x-8.$

Here we anticipate that of the two linear factors, one will have a plus sign and the other a minus. Accordingly, when we split the coefficient term, one of the parts will be negative. In this example, $2=6-4,$ giving ratios 3 : 6 and $-4:-8$. The first factor is $(x+2)$ and so the second must be $(3x-4)$.

So $3x^2+2x-8=(x+2)(3x-4)$.

Example 5: The same process applies when both the linear and the independent terms are negative, as with $3x^2-13x-10$. Splitting the middle term as $-13=-15+2$ gives the ratios $3:-15$ and $2:-10,$ leading to

$$3x^2-13x-10=(x-5)(3x+2)$$

Alternatively, it may be found easier to use Transpose and adjust and work with a positive middle term. Looking for the factors of $3x^2-13x-10.$ we then split $13=15-2,$ so that

$$3x^2-13x-10=(x+5)(3x-2)$$

Reversing the signs in these linear factors gives the factorisation $3x^2-13x-10=(x-5)(3x+2)$ as found above.

Example 6: Of course, when the x^2 coefficient is negative it is easiest to reverse all the signs and use one of the methods above. So we treat $-3x^2+13x-10$ as $(3x^2-13x+10)$ giving the factorisation $(x-5)(3x+2)$. This is another instance of *Transpose and adjust.*

Alternative splitting method: Going back to Example 2, $3x^2+10x+8,$ another way to arrive at the split for the middle term is to multiply together the first coefficient and the last, $38=24,$ and then find two factors of 24 which when added make the middle coefficient. So $24=4\times6$ and $4+6=10.$

Checking: A sub-sutra which is of great use in verifying the correctness of multiplications and factorisations is *Gunita samuccaya samuccaya gunita* or The product of the sum of the coefficients in the factors is equal to the sum of the coefficients in the product. For example, in the factorisation

$$3x^2+10x+8=(x+2)(3x+4)$$

the sums of the coefficients in the factors are respectively $1+2=3$ and $3+4=7.$ The product of the sums is $3\times7=21,$ and this is equal to the sum $3+10+8$ of the coefficients in the product, confirming the factorisation.

This can also be used as a method to help find the factors. With the quadratic $3x^2+50x+32$ the individual coefficients 3, 50, 32 offer an unhelpfully large range of possible ratios. But the sum of the factors is 85, with prime factors 5 and 17, pointing strongly to the factor $(x+16)$ and the factorisation $(3x+2)(x+16)$.

The Discriminant: It is well known that the quadratic equation Example 5

$$ax^2+bx+c=0$$

has solutions provided that the discriminant b^2-4ac is greater than or equal to zero. But what is the rational behind this formula?

To see how the discriminant arises naturally, suppose we are trying to factorise the quadratic expression. Then we seek to split the middle coefficient b into two parts b_1 and b_2 such that $a:b_1::b_2c$, or equivalently $b_1b_2=ac$. When b splits into two equal parts, $b_1=b_2=\frac{1}{2}b$, we have the largest possible product $b_1b_2=\frac{b^2}{4}$. If ac exceeds this value, no factorisation (even in surd form) will be possible. So $\frac{b^2}{4}>ac$, or $\frac{b^2}{4}\geq 0$, is a condition which must be satisfied if the quadratic ax^2+bx+c factorises, or if the equation $ax^2+bx+c=0$ has real roots.

MATHEMATICS - JEE MAIN

practice questions

1. Let $f(x)=2^{10}x+1$ and $g(x)=3^{10}x-1$, if $(fog)(x)=x$, then x is equal to

[2017]

(a) $\frac{2^{10}-1}{2^{10}-3^{-10}}$ (b) $\frac{3^{10}-1}{3^{10}-2^{-10}}$

(c) $\frac{1-2^{10}}{3^{10}-2^{-10}}$ (d) $\frac{1-3^{-10}}{2^{10}-3^{-10}}$

2. The function $f:N\to N$ is defined by

$f(x)=x-5\left[\frac{x}{5}\right]$, where N is the set of natural numbers and [x] denotes the greatest integer less than or equal to x, is

[2017]

(a) One-one but not onto

(b) One-one and onto

(c) Neither one-one nor onto

(d) Onto but not one-one

3. For $x\in R, x\neq 0, x\neq 1$, let $f_0(x)=\frac{1}{1-x}$ and $f_{n+1}(x)=f_0(f_n(X)), n=0,1,2,....$ T then the vlaue of $f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$ is equal to

[2016]

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$ (c) $\frac{5}{3}$ (d) $\frac{8}{3}$

4. Let $A=\{x_1,x_2........,x_7\}$ and $B=\{y_1,y_2,y_3\}$ be two sets containing seven and three distinct elements respectively. Then the total number of functions $f : A \to B$ that are onto, if there exist exactly three elements x in A such that $f(x) = y_2$, is equal to : **[2015]**

(a) $14.{}^7C_3$ (b) $16.{}^7C_3$

(c) $14.{}^7C_2$ (d) $12.{}^7C_2$

5. Let P be the relation defined on the set of all real numbers such that

$P = \{(a,b) : \sec^2 a - \tan^2 b = 1\}$ is: **[2014]**

(a) Reflexive and transitive but not symmetric
(b) Reflexive and symmetric but not transitive
(c) Symmetric and transitive but not reflexive
(d) An equivalence relation

6. Let f be an odd function defined on the set of real numbers such that for $x \geq 0$.

$f(x) = 3\sin x + 4\cos x$. Then $f(x)$ at $x = -\frac{11\pi}{6}$ is equal to **[2014]**

(a) $\frac{3}{2} - 2\sqrt{3}$ (b) $\frac{3}{2} + 2\sqrt{3}$

(c) $-\frac{3}{2} - 2\sqrt{3}$ (d) $-\frac{3}{2} + 2\sqrt{3}$

7. A relation on the set $A = \{x; |x| < 3, x \in z\}$, when z is the set integer is defined by

$R = \{(x, y : y = |x|, x \neq -1\}$. Then the number of elements in the power set of R is: **[2014]**

(a) 32 (b) 16 (c) 8 (d) 64

8. Let $f : R \to R$ be defined by $f(x) = \frac{|x| - 1}{|x| + 1}$ then f is: **[2014]**

(a) Both one-one and onto
(b) One-one but not onto
(c) Onto but not one-one
(d) Neither one-one nor onto

9. Let $A = \{1,2,3,4\}$ and $R : A \to A$. The correct relation defined by:

$R = \{(1,1),(2,3),(3,4),(4,2)\}$.

The correct statement is: **[2013]**

(a) R does not have an inverse
(b) R is not a one to one function
(c) R is an onto function
(d) R is not a function

10. Let $R = \{(3,3),(5,5),(9,9),(12,12),(5,12),(3,9),(3,12),$ $(3,12),(3,5)\}$ be a relation on the set $A = \{3,5,9,12\}$. Then R is **[2013]**

(a) Reflexive, symmetric but not transitive
(b) Symmetric, transitive but not reflexive
(c) An equivalence relation
(d) Reflexive, transitive but not symmetric

11. Let $R = \{(x, y) : x, y \in n \ and \ x^2 - 4xy + 3y^2 = 0\}$, where n is the set of all natural numbers. Then the relation R is: **[2013]**

(a) Reflexive but neither symmetric nor transitive
(b) Symmetric and transitive
(c) Reflexive and symmetric
(d) Reflexive and transitive

12. Consider the function:

$f(x) = [x] + |1 - x|, -1 \leq x \leq 3$ where [x] is the greatest integer funciton.

Statement 1: f is not continuous at x = 0, 1, 2 and 3.

Statement 2: $f(x) = \begin{cases} -x & -1 \leq x < 0 \\ 1 - x & 0 \leq x < 1 \\ 1 + x & 1 \leq x < 2 \\ 2 + x & 2 \leq x \leq 3 \end{cases}$ **[2013]**

(a) Statement 1 is true; Statement 2 is false,
(b) Statement 1 is true; Statement 2 is true; Statment 2 is not correct explanation for Statement 1.
(c) Statement 1 is true; Statement 2 is true; Statment it is a correct explanation for Statement 1.
(d) Statement 1 is false; Staement 2 is true.

ANSWER KEY

1. c	**2.** c	**3.** c	**4.** a
5. d	**6 .** a	**7.** b	**8.** d
9. c	**10.** d	**11.** a	**12.** a

SOLVED PAPER
JEE MAIN 2018
★ Mathematics ★

1. If the curves $y^2 = 6x, 9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :

(a) $\frac{9}{2}$ (b) 6 (c) $\frac{7}{2}$ (d) 4

2. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u} \cdot \vec{b} = 24$, then $|\vec{u}|^2$ is equal to:

(a) 84 (b) 336 (c 315 (d) 256

3. For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then

$$\lim_{x \to o^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \ldots + \left[\frac{15}{x}\right]\right)$$

(a) Does not exist (in R),
(b) Is equal to 0,
(c) Is equalto 15,
(d) Is equal to 120.

4. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0, 3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is:

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{4\sqrt{2}}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

5. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(x)}{1+2^x} dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{2}$ (d) 4π

6. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$, and $\beta(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq.units) bounded by the curve $y = (g\,o\,f)(x)$ and the lines $x = \alpha, x = \beta$ and $y = 0$, is:

(a) $\frac{1}{2}(\sqrt{2} - 1)$ (b) $\frac{1}{2}(\sqrt{3} - 1)$

(c) $\frac{1}{2}(\sqrt{3} + 1)$ (d) $\frac{1}{2}(\sqrt{3} - \sqrt{2})$

7. If the sum of all the solutions of the equation

$$8\cos(x).\left(\cos\left(\frac{\pi}{6} + x\right).\cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2}\right) = 1$$

in $[0, \pi]$ is $k\pi$, then k is equal to :

(a) $\frac{20}{9}$ (b) $\frac{2}{3}$ (c) $\frac{13}{9}$ (d) $\frac{8}{9}$

8. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}, x \in R - \{-1, 0, 1\}$

If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is :

(a) $2\sqrt{2}$ (b) 3 (c) -3 (d) $-2\sqrt{2}$

9. The integral

$$\int \frac{\sin^2(x)\cos^2(x)}{\left(\sin^5(x)+\cos^3(x)\sin^2(x)+\cos^2(x)\sin^3(x)+\cos^5(x)\right)}dx$$

(Where C is a constant of integration)

(a) $\frac{-1}{1+\cot^3(x)}+C$ (b) $\frac{1}{3(1+\tan^3(x))}+C$

(c) $\frac{-1}{3(1+\tan^3(x))}+C$ (d) $\frac{1}{1+\cot^3(x)}+C$

10. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

(a) $\frac{3}{4}$ (b) $\frac{3}{10}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

11. Let the orthocentre and centroid of a triangle be A(-3, 5) and B(3, 3) respectively. If C is the orthocentre of this triangle, then the radius of the circle having line segment AC as diameter, is

(a) $\frac{3\sqrt{5}}{2}$ (b) $\sqrt{10}$ (c) $2\sqrt{10}$ (d) $3\sqrt{\frac{5}{2}}$

12. If the tangent at (1, 7) to the curve $x^2 = y-6$ touches the circle $x^2+y^2+16x+12y+c$ then the value of c is :

(a) 95 (b) 195 (c) 185 (d) 85

13 .If $\alpha,\beta \in C$ are the distance roots , of the equation $x^2-x+1=0$, then $\alpha^{101}+\beta^{107}$ is equal to

(a) 2 (b) -1 (c) 0 (d) 1

14. PQR is a triangular park with $PQ = PR = 200m$. A T.V tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively $45°, 30°$ and $30°$, then the height of the tower (in m) is :

(a) $50\sqrt{2}$ (b) 100

(c) 50 (d) $100\sqrt{3}$

15. If $\sum_{i=1}^{9}(x_i-5)=9$ and $\sum_{i=1}^{9}(x_i-5)^2=45$, then the standard deviation of the 9 items $x_1, x_2, ..., x_9$ is:

(a) 3 (b) 9 (c) 4 (d)2

16. The sum of the co-efficients of all odd degree terms in the expansion of

$$\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5, (x>1) \text{ is:}$$

(a) 2 (b) -1 (c) 0 (d) 1

17. Tangents are drawn to the hyperbola $4x^2-y^2=36$ at the points P and Q. If these tangents intersect at the point T(0, 3) then the area(insq.units) of ΔPTQ is :

(a) $36\sqrt{5}$ (b) $45\sqrt{5}$ (c) $54\sqrt{3}$ (d) $60\sqrt{3}$

18. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is

(a) At least 750 but less than 1000

(b) At least 1000

(c) Less than 500

(d) At least 500 but less than 750

19. If the system of linear equations

$$x+ky+3z=0$$
$$3x+ky-2z=0$$
$$2x+4y-3z=0$$

has a non-zero solution (x, y, z), than $\frac{xz}{y^2}$ is equal to

(a) 30 (b) -10 (c) 10 (d) -30

20. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$, then the ordered pair (A, B) is equal to

(a) (4, 5) (b) (-4, -5) (c) (-4, 3) (d) (-4, 5)

21. Two sets A and B are as under

$$A=\{(a,b)\in R\times R : |a-5|<1 \, and \, |b-5|<1\};$$

$$B=\{(a,b)\in R\times R : 4(a-6)^2+9(b-5)^2 \le 36\},$$

Then:

(a) Neither $A \subset B$ nor $B \subset A$

(b) $B \subset A$

(c) $A \subset B$

(d) $A \cap B = \phi$ (an empty set)

22. Tangents and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$ which intersect the axis of the parabola at A and B respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan(\theta)$ is :

(a) $\frac{4}{3}$ (b) $\frac{1}{2}$ (c) 2 (d) 3

23. Let $S = \{t \in R : f(x) = |x-\pi|.(e^{|x|}-1)\sin|x|$ is not differentiable at t }. Then the set S is equal to

(a) $\{0, \pi\}$ (b) ϕ (an empty set)

(c) $\{0\}$ (d) $\{\pi\}$

24. The boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to

(a) $\sim q$ (b) $\sim p$ (c) p (d) q

25. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If Q is the origin and the rectangle $OPRQ$ is completed, then the locus of R is:

(a) $3x+2y=6xy$ (b) $3x+2y=6$

(c) $2x+3y=xy$ (d) $3x+2y=xy$

26. Let A be the sum of the first 20 terms and $\boldsymbol{B}$ be the sum of the first 40 terms of the series

$1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+..$

If $B-2A=100\lambda$, then λ is equal to:

(a) 496 (b) 232 (c) 248 (d) 464

27. Let $y=y(x)$ be the solution of the differential equation $\sin(x)\frac{dy}{dx}+y\cos(x), x\in(0,\pi)$. If $y\left(\frac{\pi}{2}=0\right)$,then $y\left(\frac{\pi}{6}\right)$ is equal to :

(a) $-\frac{4}{9}\pi^2$ (b) $\frac{4}{9\sqrt{3}}\pi^2$

(c) $-\frac{8}{9\sqrt{3}}\pi^2$ (d) $-\frac{8}{9}\pi^2$

28. The length of the projection of the line segment joining the points (5, -1, 4) and (4, -1, 3) on the plane, $x+y+z=7$ is:

(a) $\sqrt{\frac{2}{3}}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

29. Let $S = \{x \in R : x \geq 0$ and $2|\sqrt{x}-3|+\sqrt{x}(\sqrt{x}-6)+6=0\}$ then S ;

(a) Contains exactly four elements
(b) Is an empty set
(c) Contains exactly one element
(d) Contains exactly two elements

30. Let $a_1, a_2, a_3, ..., a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + ...a_{17}^2 = 140\,m$, then m is equal to;

(a)33 (b) 66 (c) 68 (d) 34

ANSWER KEY

1. a	**2.** b	**3.** d	**4.** c	**5.** a
6. b	**7.** c	**8.** a	**9.** c	**10.** c
11. d	**12.** a	**13.** d	**14.** b	**15.** d
16. a	**17.** b	**18.** b	**19.** c	**20.** d
21. c	**22.** c	**23.** b	**24.** b	**25.** d
26. c	**27.** d	**28.** a	**29.** d	**30.** d

HINTS & SOLUTIONS

1.Sol: Given $y^2 = 6x$ (1)

and $9x^2 + by^2 = 16$ (2)

slope of tangent of first curve

$$2y.\frac{dy}{dx} = 6$$

$$m_1 = \frac{dy}{dx} = \frac{6}{2y} \quad (3)$$

Slope of tangent of second curve

$$18x + 2by\frac{dy}{dx} = 0$$

$$m_2 = \frac{dy}{dx} = \frac{-18x}{2by} = \frac{-9x}{by} \quad (4)$$

Also given that curves intersects at right angle

i.e., $m_1 m_2 = -1$

$$\Rightarrow \left(\frac{6}{2y}\right)\left(-\frac{9x}{by}\right) = -1$$

$$\Rightarrow -27x = -by^2$$

$$\Rightarrow -27x = -b(6x) \quad \text{(from (1))}$$

$$\Rightarrow b = \frac{27}{6} = \frac{9}{2}$$

2.Sol: $\vec{u}\cdot\left(\vec{a}\times\vec{b}\right) = 0;$ $\quad \vec{u}\cdot\vec{a} = 0$

and $\vec{u}\cdot\vec{b} = 24$

Let $\vec{b} = \left(\vec{b}\cdot\hat{a}\right)\hat{a} + \left(\vec{b}\cdot\hat{u}\right)\hat{u}$

$$\left|\vec{b}\right|^2 = \left(\vec{b}\cdot\hat{a}\right)^2 + \left(\vec{b}\cdot\hat{u}\right)^2$$

$$\left|\vec{b}\right|^2 = \left(\vec{b}\cdot\hat{a}\right)^2 + \frac{\left(\vec{b}\cdot\vec{u}\right)^2}{\left|\vec{u}\right|^2}$$

$$2 = \frac{2}{7} + \frac{(24)^2}{\left|\vec{u}\right|^2} \Rightarrow \left|\vec{u}\right|^2 = 336$$

3.Sol : $\lim\limits_{x\to^+ x} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots+\left[\frac{15}{x}\right]\right)$

$$= \lim_{x\to 0^+} x\left(\frac{1}{x}-\left\{\frac{1}{x}\right\}+\frac{2}{x}-\left\{\frac{2}{x}\right\}+\ldots+\frac{15}{x}-\left\{\frac{15}{x}\right\}\right)$$

$$= \lim_{x\to 0^+}(1+2+3+\ldots.+15) + \lim_{x\to 0^+} x\left(\left\{\frac{1}{x}\right\}+\left\{\frac{2}{x}\right\}+\ldots.+\left\{\frac{15}{x}\right\}\right)$$

$$\left\{\therefore 0 \le \left\{\frac{r}{x}\right\} < 1, \forall x \in R\right\}$$

$$= 120$$

4.Sol: Plane passes through line of intersection of first two planes is

$$(2x-2y+3z-2)+\lambda(x-y+z+1) = 0$$

$$x(\lambda+2) - y(2+\lambda) + z(\lambda+3) + (\lambda-2) = 0 \quad (1)$$

Eq(1) have infinite number of solutions with $x+2y-z-3=0$ and $3x-y+2z-1=0$ then

$$\begin{vmatrix} (\lambda+2) & -(\lambda+2) & (\lambda+3) \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

Solving $\lambda = 5$

$$7x - 7y + 8z + 3 = 0$$

Perpendicular distance from (0, 0, 0)

is $\dfrac{3}{\sqrt{162}} = \dfrac{1}{3\sqrt{2}}$

5.Sol: Given $I = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x}dx \quad (1)$

using property $\displaystyle\int_a^b f(x)dx = \int_a^b f(a+b-x)dx,$ we have

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}}dx \quad (2)$$

adding (1) and (2)

$$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$$

$$\Rightarrow 2I = 2.\int_0^{\pi/2} \sin^2 x\, dx$$

$$\Rightarrow 2I = 2\times\frac{\pi}{4} \Rightarrow I = \frac{\pi}{4}$$

6.Sol: $18x^2 - 9\pi x + \pi^2 = 0$

$$(6x-\pi)(3x-\pi) = 0$$

i.e., $x = \dfrac{\pi}{6}, \dfrac{\pi}{3}$

$$\therefore \quad \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$$

now, $y = (gof)(x) = \cos x$

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\, dx = \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$= \frac{1}{2}(\sqrt{3}-1)\,sq.units$$

7.Sol : $8\cos x.\left[\left(\cos^2\frac{\pi}{6}-\sin^2 x\right)-\frac{1}{2}\right]=1$

$\Rightarrow \quad 8\cos x\left(\frac{3}{4}-\frac{1}{2}-1+\cos^2 x\right)=1$

$\Rightarrow \quad \frac{8\cos x}{4}\times\left(4\cos^2 x-1-2\right)=1$

$\Rightarrow \quad \cos 3x=4\cos^3 x-3\cos x$

$\Rightarrow \quad 2\times\cos 3x=1$

$\Rightarrow \quad \cos 3x=\frac{1}{2}$

i.e., $\quad 3x\in[0,3\pi]$

$\therefore \quad 3x=\frac{\pi}{3},2\pi-\frac{\pi}{3},2\pi+\frac{\pi}{3}$

$\therefore \quad \text{sum}=\frac{13\pi}{9}$

8.Sol: Let $g(x)=x-\frac{1}{x}=t$

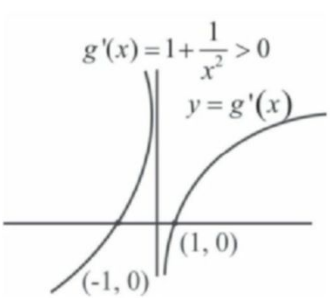
$g'(x)=1+\frac{1}{x^2}>0$

$\Rightarrow \quad t\in R-\{0\};t^2\in(0,\infty)$

Now, $f(x)=x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2=t^2+2\in(2,\infty)$

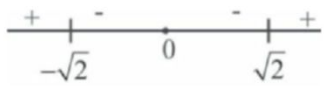
Now, $\quad h(x)=\frac{f(x)}{g(x)}$

$\therefore \frac{f(x)}{g(x)}=\frac{t^2+2}{t}=t+\frac{2}{t}$



Let $\quad h(t)=t+\frac{2}{t}$

$h'(t)=1-\frac{2}{t^2}$



$\therefore$ Local minimum value occurs at $t=\sqrt{2}$

$\therefore$ The local minimum value is $h\left(\sqrt{2}\right)=2\sqrt{2}$

9.Sol: Let $I=\int\frac{\sin^2 x\cos^2 x \quad dx}{\left[\left(\sin^2 x+\cos^2 x\right)\left(\sin^3 x+\cos^3 x\right)\right]^2}$

$=\int\frac{\sin^2 x\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2}dx$

$=\int\frac{\tan^2 x.\sec^2 x}{\left(1+\tan^3 x\right)^3}dx$

Put $\quad (1+\tan^3 x)=t$

$3\tan^2 x\sec^2 x\,dx=dt$

$I=\frac{1}{3}\int\frac{dt}{t^2}=-\frac{1}{3t}+C$

Hence, $\quad I=\frac{-1}{3(1+\tan^3 x)}+C$

10.Sol : E_1 : Event that first ball drawn is red

E_2 : Event that first ball drawn is black

E_3 : Event that second ball drawn is red.

$P(E)=P(E_1).P\left(\frac{E}{E_1}\right)+P(E_2).P\left(\frac{E}{E_2}\right)$

$=\frac{4}{10}\times\frac{6}{12}+\frac{6}{10}\times\frac{4}{12}=\frac{2}{5}$

11.Sol: Orthocentre $A(-3, 5)$

Centroid $B(3, 3)$

By section formula we have

$\Rightarrow \quad \frac{2h+(-3)}{3}=3$

$\Rightarrow \quad 2h-3=9$

$\Rightarrow \quad h = 6$

Now, $\dfrac{2k+5}{3} = 3$

$\Rightarrow \quad 2k+5 = 9$

$\Rightarrow \quad k = 2$

$\Rightarrow \quad A(-3, 5)$

Diameter $AC = \sqrt{9^2+3^2} = 3\sqrt{9+1} = 3\sqrt{10}$

Radius $= \dfrac{3\sqrt{10}}{2} = 3\sqrt{\dfrac{5}{2}}$

12.Sol: Given curve is $x^2 = y-6$

$\Rightarrow \quad 2x = \dfrac{dy}{dx}$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(1,7)} = 2$

Equation of tangents at (1, 7) is

$y-7 = 2(x-1)$

$\Rightarrow 2x - y + 5 = 0 \qquad (1)$

Line (1) touches the circle

$x^2+y^2+16x+12y+c=0$

$\therefore$ Perpendicular distance from the centre of the circle is equal to radius of the circle

i.e., $\left|\dfrac{2(-8)-(-6)+5}{\sqrt{4+1}}\right| = \sqrt{64+36-c}$

$\Rightarrow \quad \dfrac{25}{5} = 100 - c$

$\Rightarrow \quad 5 = 100 - c$

$\Rightarrow c = 95$

13.Sol : Given α, β are the distinct roots of the equation $x^2 - x + 1 = 0$

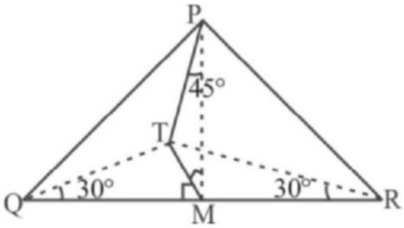
i.e., $x = \dfrac{1 \pm \sqrt{-3}}{2} = -\omega, -\omega^2$

(where ω and ω^2 are non-real cube roots of unity)

$\Rightarrow \alpha = -\omega$ and $\beta = -\omega^2$

$\Rightarrow (-\omega)^{101} + (-\omega)^{107} = -\left(\omega^{101} + \omega^{214}\right) = -\left(\omega^2 + \omega\right) = 1$

14.Sol:



Let height of tower TM be n

$\therefore PM = n$

In Δ TQM, $\tan 30° = \dfrac{h}{QM}$

$QM = \sqrt{3}h$

In Δ PMQ, $PM^2 + QM^2 = PQ^2$

$h^2 + \left(\sqrt{3}h\right)^2 = 200^2$

$\Rightarrow \quad 4h^2 = 200^2$

$\Rightarrow \quad h = 100m$

15.Sol: Given $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i-5)^2 = 45$

$\sum x_i = 54$ and $\sum x_i^2 - 10\sum x_i + 9(25) = 45$

i.e., $\sum x_i^2 - 10(54) + 225 = 45$

$\Rightarrow \quad \sum x_i^2 = 360$

Now variance $= \dfrac{\sum x_i^2}{9} - \left(\dfrac{\sum x_i^2}{9}\right)$

$= \dfrac{360}{9} - \left(\dfrac{54}{9}\right)^2$

$= 40 - 36 = 4$

Hence standard deviation is 2

16.Sol: we know $(x+a)^5 + (x-a)^5$

$= 2\left[{}^5C_0 x^5 + {}^5C_2 x^3.a^2 + {}^5C_4 x.a^4\right]$

i.e., $\left(x+\sqrt{x^3-1}\right)^5 + \left(x - \sqrt{x^3-1}\right)^5$

$= 2\left[{}^5C_0 x^5 + {}^5C_2 x^3\left(x^3-1\right) + {}^5C_4 x(x^3-1)^2\right]$

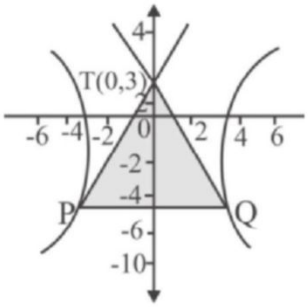
$= 2\left[x^5 + 10x^6 - 10x^3 + 5x^7 - 10x^4 + 5x\right]$

Now consider odd degree terms

$$2\left[x^5+5x^7-10x^3+5x\right]$$

Sum of coefficient of odd terms is 2

17.Sol: Hyperbola $\frac{x^2}{9}-\frac{y^2}{36}=1$



PQ is a chord of contact w.r.t $T\,(0,\,3)$

Equation of PQ is $T=0$

i.e., $\frac{x(0)}{9}-\frac{y(3)}{36}=1$

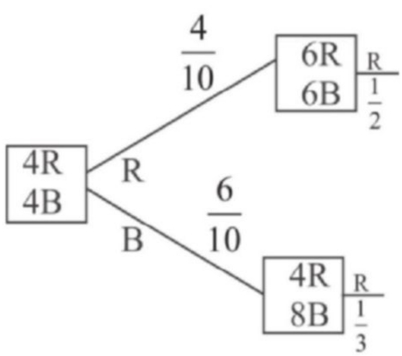
$\Rightarrow \quad y=-12$

$y=-12$ intersect the hyperbola at P & Q

$\therefore P\left(-\sqrt{45},-12\right);\quad Q(\sqrt{45},-12)$

Area of $\Delta PTQ=\frac{1}{2}\begin{vmatrix}-\sqrt{45} & -12 & 1\\ 0 & 3 & 1\\ \sqrt{45} & -12 & 1\end{vmatrix}=45\sqrt{5}$

18.Sol:



Number of ways of selecting 4 novels from 6 novels $={}^6C_4$

Number of ways of selecting 1 dictionary from 3 dictionaries $={}^3C_1$

Required arrangements $={}^6C_4\times{}^3C_4\times 4!=1080$

Atleast 1000

19.Sol: Given $\begin{vmatrix}1 & k & 3\\ 3 & k & -2\\ 2 & 4 & -3\end{vmatrix}=0$

$$k=\frac{7}{2}$$

$x+ky+3z=0 \qquad (1)$

$3x+ky-2z=0 \qquad (2)$

$2x+4y-3z=0 \qquad (3)$

On solving(1) and (2)

$2x-5z=0 \qquad (4)$

On solving (3) and (4) $4y=-2z$, we get

$$\frac{xz}{y^2}=\frac{\frac{5}{2}z\times z}{\frac{z^2}{4}}=10$$

20.Sol: now, $\begin{vmatrix}x-4 & 2x & 2x\\ 2x & x-4 & 2x\\ 2x & 2x & x-4\end{vmatrix}=(A+Bx)(x-A)^2$

Put $x=0\Rightarrow\begin{vmatrix}-4 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{vmatrix}=A^3\Rightarrow A=-4$

now, $\begin{vmatrix}x-4 & 2x & 2x\\ 2x & x-4 & 2x\\ 2x & 2x & x-4\end{vmatrix}=(Bx-4)(x+4)^2$

$\frac{R_1}{x}\to R_1;\frac{R_2}{x}\to R_2$ and $\frac{R_3}{x}\to R_3$

$$\begin{vmatrix}1-\frac{4}{x} & 2 & 2\\ 2 & 1-\frac{4}{x} & 2\\ 2 & 2 & 1-\frac{4}{x}\end{vmatrix}=\left(B-\frac{4}{x}\right)\left(1+\frac{4}{x}\right)^2$$

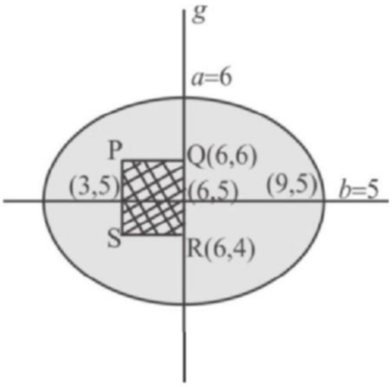
as $x \to \infty \Rightarrow \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = B \Rightarrow B = 5$

The desired order pair is $(-4, 5)$.

21.Sol: Given $|a-5| < 1$ and $|b-5| < 1$

i.e., $4 < a, b < 6$ and $\frac{(a-6)^2}{9} + \frac{(b-5)^2}{4} \le 1$

Taking axes as a-axis and b-axis



Hence, the square that represent the set A, like wise the ellipse represents the set B.
Clearly from the above graph, we see the set A is inside the set B. i.e., $A \subset B$

22.Sol: $y^2 = 16x$

Slope of tangent $\frac{dy}{dx} = \frac{16}{2y} = \frac{8}{y}$

$$\left(\frac{dy}{dx}\right)_{(16,16)} = \frac{1}{2}$$

Equation of tangent $y - 16 = \frac{1}{2}(x - 16)$

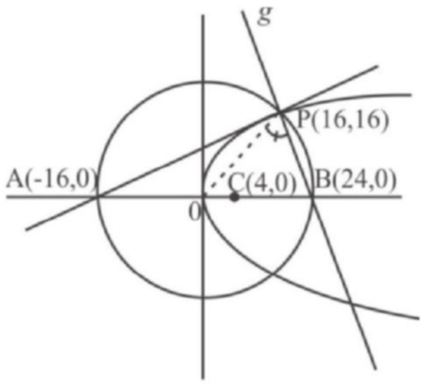
$\Rightarrow \quad x - 2y + 16 = 0 \qquad (1)$

Equation of normal is $y - 16 = -2(x - 16)$

$2x + y - 48 = 0 \qquad (2)$

Given that, Tangent & normal intersect the axis of parabola i.e.,

$A(-16, 0) \qquad B(24, 0)$



We know product of slope of a tangent and normal is -1

i.e., $\left(\frac{16-0}{16+16}\right)\left(\frac{16-0}{16-24}\right) = -1$

$\therefore AB$ is diameter of circle i.e. centre $C(4, 0)$

$$m_{PC} = \frac{16-0}{16-4} = \frac{4}{3}$$

$$m_{PB} = \frac{16-0}{16-24} = -2$$

$$\tan\theta = \left|\frac{\frac{3}{4} - (-2)}{1 + \left(\frac{4}{3}\right)(-2)}\right| = 2$$

23.Sol: Given $f(x) = |x - \pi|\left(e^{|x|} - 1\right)\sin|x|$

We check differentiable at $x = \pi$ & $x = 0$
Now at $x = \pi$, we have

$$\text{R.H.D} = \lim_{h \to 0^+} \frac{|\pi + h - \pi|\left(e^{|\pi+h|} - 1\right)\sin(\pi + h) - 0}{h} = 0$$

$$\text{L.H.D} = \lim_{h \to 0^+} \frac{|\pi - h - \pi|\left(e^{|\pi-h|} - 1\right)\sin(\pi - h) - 0}{-h} = 0$$

$\therefore$ R.H.D=L.H.D, so function is differentiable at $x = \pi$

Like wise at $x = 0$, we have

$$\text{R.H.D} = \lim_{h \to 0^+} \frac{|h - \pi|\left(e^{(h)} - 1\right)\sin|h| - 0}{h} = 0$$

L.H.D $= \lim_{h\to 0^+} \dfrac{|-h-\pi|\left(e^{(-h)}-1\right)\sin|-h|-0}{-h} = 0$

$\therefore$ RHD=LHD, so function is differentiable at $x=0$

Set S is empty set, i.e., ϕ.

24.Sol: $\sim(p \vee q) \vee (\sim p \wedge q)$

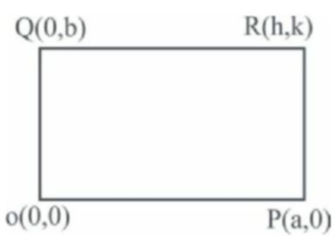
P	q	$\sim(P\vee q)$	$\sim P\wedge q$	$\sim P$
T	F	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	T	F	T

25.Sol: Let the equation of line be $\dfrac{x}{a}+\dfrac{y}{b}=1$ (1)

Given that, eq (1) passes through the fixed point (2,3)

i.e., $\dfrac{2}{a}+\dfrac{3}{b}=1$ (2)

$P(a,0), Q(0,b), O(0,0), R(h,k)$



Mid point OR is $\left(\dfrac{h}{2},\dfrac{k}{2}\right)$

Midpoint of PQ is $\left(\dfrac{a}{2},\dfrac{b}{2}\right) \Rightarrow h=a, k=b$ (3)

from (2) & (3), we get

$$\frac{2}{h}+\frac{3}{k}=1 \Rightarrow \text{locus of } R(h, k)$$

$$\Rightarrow \quad \frac{2}{x}+\frac{3}{y}=1 \Rightarrow 3x+2y=x$$

26.Sol: Sum of first 20 terms

$$A=\left(1^2+3^2+....+19^2\right)+2\left(2^2+4^2+....+20^2\right)$$

$$A=\left(1^2+2^2+3^2+....+19^2+20^2\right)+\left(2^2+4^2+....+20^2\right)$$

$$\Rightarrow \quad A=\frac{20\times 41\times 21}{6}+2^2\left(1^2+2^2+....+10^2\right)$$

$$\Rightarrow \quad A=2870+4\times\frac{10\times 11\times 21}{6}$$

i.e., $A=4410$

Now, sum of first 40 terms

$$B=\left(1^2+2^2+....40^2\right)+\left(2^2+4^2+....+40^2\right)$$

$$=\frac{40\times 41\times 81}{6}+2^2\left(1^2+2^2+...+20^2\right)$$

$$=\frac{40\times 41\times 81}{6}+4\left(\frac{20\times 21\times 41}{6}\right)$$

$$=33620$$

Given $B-2A=100\lambda$

i.e., $33620-8820=100\lambda$

$24800=100\lambda$

$\lambda=248$

27.Sol: $\sin x\dfrac{dy}{dx}+y.\cos x=4x$

$$\Rightarrow \frac{dy}{dx}+y.\cot x=\frac{4x}{\sin x}, x\neq 0$$

Now integrating factor is

$$I\cdot F=e^{\int \cot x\, dx}=e^{\log\sin x}=\sin x$$

Multiplying integration factor, we get

$$y.\sin x=\int\frac{4x}{\sin x}.\sin x\, dx$$

$$y.\sin x=\int 4x\, dx$$

$$y.\sin x=2x^2+c \quad (1)$$

at $x=\dfrac{\pi}{2}, y=0$

$$\therefore c=-\frac{\pi^2}{2}$$

From (1) $\Rightarrow y.\sin x=2x^2-\dfrac{\pi^2}{2}$

at $x = \frac{\pi}{6}$, we have

$$y.\frac{1}{2} = 2 \times \frac{\pi^2}{36} - \frac{\pi^2}{2}$$

$$y = \frac{\pi^2}{9} - \pi^2$$

$$= -\frac{8\pi^2}{9}$$

28.Sol: $\frac{x-5}{1} = \frac{y+1}{1} = \frac{z-4}{1} = \lambda$

$$P(\lambda+5, \lambda-1, \lambda+4)$$

P is foot of perpendicular from A to plane

$3\lambda + 8 = 7$

$\Rightarrow \quad \lambda = -\frac{1}{3}$

Now, $\quad P\left(\frac{14}{3}, \frac{-4}{3}, \frac{11}{3}\right)$

$$\frac{x-4}{1} = \frac{y+1}{1} = \frac{z-3}{1}$$

$$Q(\lambda+4, \lambda-1, \lambda+3)$$

Q is foot of perpendicular from B to plane $3\lambda+6=7$

$\Rightarrow \quad \lambda = \frac{1}{3}$

now $\quad Q\left(\frac{13}{3}, \frac{-2}{3}, \frac{10}{3}\right)$

$$\therefore PQ = \frac{\sqrt{1+4+1}}{3} = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$$

29.Sol: **Case-1:** $x \in [10, 9]$

$$2(3-\sqrt{x}) + x - 6\sqrt{x} + 6 = 0$$

$\Rightarrow \quad x - 8\sqrt{x} + 12 = 0 \Rightarrow \sqrt{x} = 4, 2$

$$x = 16, 4 \Rightarrow x = 4 \ \{\because 16 \notin [0, 9]\}$$

Case-II: $\quad x \in [9, \infty]$

$$2\left(\sqrt{x} - 3\right) + x - 6\sqrt{x} + 6 = 0$$

$$x - 4\sqrt{x} = 0$$

$$\Rightarrow x = 0, 16$$

$$\therefore x = 16 \{\because 0 \notin [9, \infty]\}$$

So $\quad x = 4, 16$

30.Sol: Let $a_1 = a$ and common difference $= d$

Given, $a_1 + a_5 + a_9 + ... + a_{49} = 416$

$\Rightarrow a + 24d = 32 \qquad (1)$

Also, $a_9 + a_{43} = 66 \Rightarrow a + 25d = 33 \qquad (2)$

Solving (1) & (2)

We get $d = 1, a = 8$

Now, $a_1^{\ 2} + a_2^{\ 2} + ... + a_{17}^{\ 2} = 140m$

$$\Rightarrow 8^2 + 9^2 + ... + 24^2 = 140m$$

$$\Rightarrow \frac{24 \times 25 \times 49}{6} - \frac{7 \times 8 \times 15}{6} = 140m$$

$$\Rightarrow m = 34$$

Challenging Problems JEE

By: **Rajan L Shodhan**(Ahemdabad)

1. The value of $e^{2019 i \cot^{-1} p} \cdot \left[\frac{pi+1}{pi-1}\right]^{1009\cdot 5}$ is equal to

(a) 1 (b) 0 (c) -1 (d) $\frac{1}{e}$

2. If $f(x) = \cos x - \cos^2 x + \cos^3 x \ldots \infty$, then

$\int_0^{2019\frac{\pi}{2}} f(x)dx$ is equal to

(a) 1 (b) $\frac{2019\pi}{2}+1$

(c) $\frac{2019\pi}{2}-1$ (d) -1

3. If α and β are the roots of $x^2 - x + 1 = 0$, then $\alpha^{2019} + \beta^{2019}$ is equal to

(a) 0 (b) 1 (c) 2 (d) -2

4. Let $f(k) = \frac{k}{2019}$ and $g(k) = \frac{f^4(k)}{[1-f(k)]^4 + [f(k)]^4}$

then the sum of $\sum_{k=0}^{2019} g(k)$ is equal to

(a) 2019 (b) 2008 (c) 1010 (d) 1009

5. Let, $f: I^+ \to R$ be a function such that $\forall n > 1, \sum_{r=1}^{n} r \cdot f(r) = n \cdot (n+1) f(n)$ and $f(1) = 1$, the value of $f(1009.5)$ will be

(a) $\frac{1}{2018}$ (b) $\frac{1}{2019}$

(c) $\frac{1}{2019 \cdot 5}$ (d) $2018 \cdot 5$

6. A function 'f' satisfies the relation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in N$ and $f(1) = 2$. If $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where $a \in N$, then $673a$ is

(a) 1346 (b) 2692

(c) 2019 (d) $\frac{-1}{2019}$

7. If $f(x)$ is a function such that $f(x-1) + f(x+1) = \sqrt{3} \cdot f(x)$ and $f(5) = 3$

then $\sum_{r=0}^{672} f(5+12r)$ is equal to

(a) 2019 (b) 673

(c) 1 (d) $672 \cdot \sqrt{3}$

8. Consider the sequence 1,2,2,3,3,3,4,4,4,4,... find the 2019^{th} term of the sequence. Find also the sum of first 2019 terms.

(a) 62 and 84396 (b) 63 and 85,533
(c) 63 and 84396 (d) 62 and 85,533

9. The remainder, when 2^{2019} is divided by 63, is equal to

(a) 2 (b) 4 (c) 6 (d) 8

10. Total number of real values of x, such that

$$\frac{(2019+x)^{\frac{1}{7}}}{x}+\frac{(2019+x)^{\frac{1}{7}}}{2019}=\frac{2187}{673}\times\frac{1}{7} \text{ is equal to}$$

(a) $\frac{2186}{2019}$ (b) $\frac{2019}{2186}$

(c) $\frac{673}{2019}$ (d) $\frac{2019}{2187}$

ANSWER KEY

1. a	2. b	3. d	4. c	5. b
6. c	7. a	8. b	9. d	10.b

HINTS & SOLUTIONS

1.Sol: $e^{2019i\cdot\cot^{-1}p}\cdot\left[\frac{pi+1}{pi-1}\right]^{1009\cdot5}$

To put $\cot^{-1}p=\theta\Rightarrow\cot\theta=p$ and $i=\sqrt{-1}$

$\Rightarrow i^2=-1$

$$\therefore\quad e^{2019i\theta}\left[\frac{i\cot\theta-i^2}{i\cot\theta+i^2}\right]^{1009\cdot5}$$

$$=e^{2019i\theta}\left[\frac{i(\cos\theta-i\sin\theta)}{i(\cos\theta)+i\sin\theta}\right]^{1009\cdot5}$$

$$=e^{2019i\theta}\left[\frac{e^{-i\theta}}{e^{i\theta}}\right]^{1009\cdot5}$$

$$=e^{2019i\theta}\cdot\left[e^{-2i\theta}\right]^{1009\cdot5}$$

$$=e^{2019i\theta-2019i\theta}=e^0=1$$

2.Sol: Given $f(x)=\cos x-\cos^3 x+\cos^3 x\ldots\infty$

$$=\frac{\cos x}{1+\cos x}\quad\left(\because S_\infty=\frac{9}{1-r}\right)$$

$$\therefore\quad f(x)=\frac{\cos x}{1+\cos x}$$

$$\therefore\quad\int_0^{2019\frac{\pi}{2}}\frac{\cos x}{I+\cos x}dx\Rightarrow\int_0^{2019\frac{\pi}{2}}\frac{1+\cos x-1}{1+\cos x}dx$$

$$=\int_0^{2019\frac{\pi}{2}}dx-\int_0^{2019\frac{\pi}{2}}\frac{1}{2\cos^2\frac{x}{2}}dx$$

$$=[x]_0^{2019\frac{\pi}{2}}-\frac{1}{2}\int_0^{2109\frac{\pi}{2}}\sec^2\frac{x}{2}dx$$

$$=2019\frac{\pi}{2}-\frac{1}{2}\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{2019\frac{\pi}{2}}$$

$$=2019\frac{\pi}{2}-\left[\tan 2019\frac{\pi}{2}-0\right]$$

$$=2019\frac{\pi}{2}-\tan\left[505\pi-\frac{\pi}{4}\right]$$

$$=2019\frac{\pi}{2}+\tan\frac{\pi}{4}$$

$$=2019\frac{\pi}{2}+1$$

3.Sol: Let $x^2-x+1=0$

$$\Rightarrow\quad x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm i\sqrt{3}}{2}$$

$\therefore\quad x=-\omega$ and $\beta=-\omega^2$

Now, $\alpha^{2019}+\beta^{2019}=(-\omega)^{2019}+(-\omega^2)^{2019}$

$$=(-1)^{2019}(\omega^3)^{673}+(-1)^{2019}(\omega^3)^{1346}$$

$$= -1-1 \qquad (\because \omega^3 = 1)$$

4.Sol: Let $f(k) = \dfrac{k}{2019} \Rightarrow f(2019-k) = \dfrac{2019-k}{2019}$

$$\Rightarrow f(k) + f(2019-k) = \frac{k}{2019} + \frac{2019-k}{2019} = 1$$

$$\therefore \quad f(k) + f(2019-k) = 1 \qquad (1)$$

Again, $$g(k) = \frac{f^4(k)}{[1-f(k)]^4 + [f(k)]^4} \qquad (2)$$

$$\Rightarrow g(2019-k) = \frac{[f(2019)-k]^4}{[1-f(2019-k)]^4 + [f(2019-k)]^4}$$

$$= \frac{[1-f(k)]^4}{[f(k)]^4 + [1-f(k)]^4} \qquad (3)$$

$(\because$ from (1), $f(k) + f(2019-k) = 1$

Take (2)+(3), we get

$$g(k) + g(2019-k)$$

$$= \frac{f^4(k)}{[1+f(k)]^4 + f^4(k)} + \frac{[1-f(k)]^4}{f^4(k) + [1-f(k)]^4}$$

$$= \frac{f^4(k) + [1-f(k)]^4}{f^4(k) + [1-f(k)]^4} = 1$$

$$\therefore g(k) + g(2019-k) = 1$$

$$\therefore \quad g(0) + g(2019) = 1$$

$$g(1) + g(2018) = 1$$

$$g(2) + g(2017) = 1$$

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$$g(1009) + g(1010) = 1$$

Adding them, we have

$$\sum_{k=0}^{2019} g(k) = 1+1 = 1+ \ldots 1010 \text{ times}$$

$$\sum_{k=0}^{2019} g(k) = 1010$$

5.Sol: $\sum_{r=1}^{n} r \cdot f(r) = n \cdot (n+1) \cdot f(n)$

$$\Rightarrow f(1) + 2 \cdot f(2) + 3 \cdot f(3) + \ldots + n \cdot f(n) = n \cdot (n+1) \cdot f(n) \quad (1)$$

now we replace by $(n+1)$, we get

$$f(1) + 2 \cdot f(2) + 3 \cdot f(3) + \ldots + n \cdot f(n) + (n+1) \cdot f(n+1)$$

$$= (n+1)(n+2) \cdot f(n+1) \qquad (2)$$

Take (2) - (1) we get

$$(n+1) \cdot f(n+1) = (n+1) \cdot (n+2) \cdot f(n+1) - n \cdot (n+1) \cdot f(n)$$

$$\Rightarrow n \cdot (n+1) \cdot f(n) = (n+1) \cdot f(n+1)[n+2-1]$$

$$\Rightarrow n \cdot (n+1) \cdot f(n) = (n+1)^2 \cdot f(n+1)$$

$$\therefore \quad n \cdot f(n) = (n+1) \cdot f(n+1)$$

$$\therefore \quad 2 \cdot f(2) = 3 \cdot f(3) = 4 \cdot f(4) = n \cdot f(n)$$

Put in equation (1), we have

$$f(1) + [n \cdot f(n) + n \cdot f(n) + \ldots (n-1) \text{times}]$$

$$= n \cdot (1+n) \cdot f(n)$$

$$\therefore \quad f(1) + (n-1) \cdot n \cdot f(n) = n \cdot (1+n) \cdot f(n)$$

$$\therefore \quad f(1) = n \cdot f(n)[n+1-n+1]$$

$$\therefore \quad f(1) = 2n \cdot f(n)$$

$$\therefore \quad f(n) = \frac{f(1)}{2n} = \frac{1}{2n} \qquad (\because f(1) = 1)$$

$$\therefore \quad f(n) = \frac{1}{2n}$$

$$\Rightarrow \quad f(1009 \cdot 5) = \frac{1}{2(1009 \cdot 5)} = \frac{1}{2019}$$

$$\therefore \quad f(1009 \cdot 5) = \frac{1}{2019}$$

6.Sol: Let, $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$$\Rightarrow f(a+1) + f(a+2) + f(a+3) + \ldots + f(a+n)$$

$$= 16[2^n - 1]$$

Now $f(x+y)=f(x)\cdot f(y)$

$\Rightarrow f(a+1)=f(a)\cdot f(1)=2\cdot f(a)$

$f(a+2)=f(a)\cdot f(2)=2^2\cdot f(a)$

$f(a+3)=f(a)\cdot f(3)=2^3\cdot f(a)$

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$f(a+n)=f(a)\cdot f(n)=2^n\cdot f(a)$

Adding them

$$\sum_{k=1}^{n} f(a+k)=f(a)\left[f(1)+f(2)+...+f(n)\right]$$

$$=16\left[2^n-1\right]$$

$$=f(a)\left[2+2^2+2^3+...+2^n\right]=16\left[2^n-1\right]$$

$[\therefore\ f(1)=2, f(2)=f(1+1)=f(1)\cdot f(1)=2\cdot 2=2^2,$

$f(3)=f(2+1)=f(2)f(1)=2^2\cdot 2=2^3...]$

$$=f(a)\left[\frac{2(2^n-1)}{2-1}\right]=16(2^n-1)$$

$\Rightarrow\ f(a)=8$

$\therefore\ f(1)=2, f(2)=2^2, f(3)=2^3, f(4)=2^4,...f(4)=2^9,$

$\therefore\ 2^9=2^3 \qquad (\because f(a)=8=2^3)$

$\therefore\ a=3$

$\therefore (673)a=673\times 3=2019$

$\therefore\ (673)a=2019$

7.Sol: $f(x-1)+f(x+1)=\sqrt{3}\cdot f(x) \qquad (1)$

replace x be $(x+2)$

$\therefore\ f(x+1)+f(x+3)=\sqrt{3}\cdot f(x+2) \qquad (2)$

Take (1)+(2), we get

$$f(x-1)+f(x+3)+2\cdot f(x+1)=\sqrt{3}\left[f(x)+f(x+2)\right]$$

$$=\sqrt{3}\cdot\left[\sqrt{3}\cdot f(x+1)\right]$$

($\therefore$ from (1))

$$f(x-1)+f(x+3)=3\cdot f(x+1)-2\cdot f(x+1)$$

$\therefore\ f(x-1)+f(x+3)=f(x+1) \qquad (3)$

Again replace 'x' by $x+2$ in (3), we have

$f(x+1)+f(x+5)=f(x+3) \qquad (4)$

Take (3) + (4), we have

$$f(x-1)+f(x+3)+f(x+1)+f(x+5)=f(x+1)+f(x+3)$$

$\therefore\ f(x-1)=-f(x+5)$

To put $x=x+1$

$\therefore\ f(x)=-f(x+6) \qquad (5)$

$\therefore\ f(x+12)=f(x+6+6)=f(x+6)\cdot f(6)$

$=-f(x)\cdot f(6)$

(from (5))

$=-f(x+6)$

$=f(x)$

$\therefore\ f(x+12)=f(x)$

$$\therefore\ \sum_{r=0}^{672} f(5+12r)$$

$$=f(5)+f(5+12)+f(5+2\cdot 12)+f(5+3\cdot 12)$$

$$+...+f(5+672(12))$$

$=f(5)+f(5)+f(5)+...$ upto 673 times

$$(\because f(5+12)=f(5), f(5+2\cdot 12)=f(5+12+12))$$

$$=f(5+12)\cdot f(12)=f(5)\cdot f(12)=f(5+12)=f(5)$$

$=673\times f(5)$

$=673\times 3=2019$

$$\therefore\ \sum_{r=0}^{672} f(5+12)=2019$$

8.Sol: Let 1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,.........

Let us write the terms in groups as under

1,(2,2), (3,3,3), (4,4,4,4), (5,5,5,5,5).....

Consisting 1,2,3,4,5....terms

Let 2019^{th} terms falls in n^{th} group.

$$\Rightarrow\ \frac{n(n-1)}{2}<2019\le\frac{n(n+1)}{2}$$

$$\Rightarrow\ n(n-1)<4038\le n(n+1)$$

Let us consider, $n(n-1) < 4038$

$$\Rightarrow \quad n^2 - n - 4038 < 0$$

$$\Rightarrow \quad n < \frac{1+\sqrt{1+16152}}{2}$$

$$\Rightarrow \quad n < \frac{1+\sqrt{16153}}{2}$$

$$\Rightarrow \quad n < \frac{1+127\cdot 1}{2} \Rightarrow n < \frac{128\cdot 1}{2}$$

$$\Rightarrow \quad n < 64$$

For $n(n+1) \geq 4038 \Rightarrow n^2 + n - 4038 \geq 0$

$$\therefore \quad n \geq \frac{-1+\sqrt{16153}}{2}$$

$$\Rightarrow \quad n \geq \frac{-1+127\cdot 1}{2} \Rightarrow n \geq \frac{126\cdot 1}{2}$$

$$\Rightarrow \quad n \geq 63$$

$$\therefore \quad n = 63$$

That means 2019^{th} terms falls in 63^{rd} group

Now total number of terms upto 62^{th} group

$$= \frac{62\times 63}{2} = 31\times 63 = 1953$$

$$\Rightarrow 1^2 + 2^2 + 3^2 + .. + 62^2 + 63[2019-1953]$$

$$(\because 1, (2,2) = 2+2 = 4 = 2^2, (3,3,3) = 3+3+3 = 9 = 3^2,$$
$$(4,4,4,4) = 4^2, ...)$$

$$= \frac{62\times 63\times 125}{6} + (63\times 66)$$

$$= 85,533$$

$\therefore$ 2019^{th} term of the sequence is 63 and the sum of first 2019 term is 85,533

9.Sol: Let $2^{2019} = \left(2^6\right)^{336}\cdot 2^3$

$$= (64)^{336}\cdot 8$$

$$= [63+1]^{336}\cdot 8$$

$$= \left[336C_0\cdot 63^{336} + 336C_1\cdot(63)^{335} + 336C_2(63)^{334} + \right.$$
$$\left. ... + 336C_{336}(7)^{336}\right]\cdot 8$$

$$= (63\times 8)\left[(63)^{335} + 336C_1(63)^{334} + 336C_2(63)^{333} \right.$$
$$\left. + ..\right] + 8$$

$= 63m + 8$, where $m \in I$

$\therefore$ When 2^{2019} is divided by 63, we get remainder is 8

10.Sol: Let,

$$\frac{(2019+x)^{\frac{1}{7}}}{x} + \frac{(2019+x)^{\frac{1}{7}}}{2019} = \frac{2187}{673}\times\frac{1}{7}$$

$$\Rightarrow \quad (2019+x)^{\frac{1}{7}}\left[\frac{2019+x}{(2019)(x)}\right] = \frac{2187}{673}\times\frac{1}{7}$$

$$\Rightarrow \quad (2019+x)^{\frac{1}{7}+1} = 3\cdot 3^7 x^{1+\frac{1}{7}}$$

$$\therefore \quad (2019+x)^{\frac{8}{7}} = 3^8\cdot x^{\frac{8}{7}}$$

$$\Rightarrow \quad 2019 + x = 3^7\cdot x$$

$$\Rightarrow \quad 2019 + x = 2187x$$

$$\Rightarrow \quad 2186x = 2019$$

$$\therefore \quad x = \frac{2019}{2186}$$

EVAAITS (JEE ADVANCED - 3) SOLUTIONS

ANSWER KEY

Section I:

1. 3 **2.** 4 **3.** 4 **4.** 4 **5.** 7
6. 0 **7.** 5 **8.** 5

Section II:

1. a,b,c,d **2.** a,c **3.** a,c **4.** a,b,d **5.** a
6. b,c **7.** a,b **8.** a,c

Section III:

1. c **2.** b **3.** b **4.** c

HINTS & SOLUTIONS

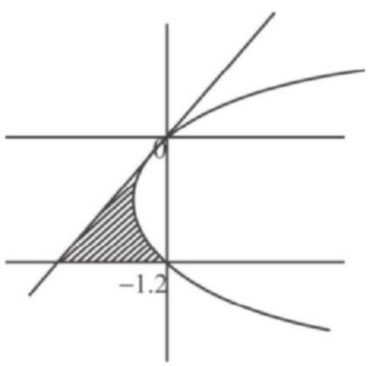
Section I:

1.Sol: $y^2 + y = x$ is plotted as in adjacent diagram

$$\Rightarrow \quad 2y\frac{dy}{dx} + \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y+1}$$

Now $\left.\frac{dy}{dx}\right|_{(0,0)} = 1$



$\therefore$ Equation of tangent at origin is $y = x$

Now the desired area is shaded area in the above graph.

i.e., $\Delta = \int_{-1}^{0} \left[\left(y^2 + y\right) - y\right] dy = \frac{1}{3}$

2.Sol: $x\frac{dy}{dx} - \frac{x^2}{y} = -y^3$

Now, put $x^2 = t$

$$\Rightarrow \quad \frac{dt}{dy} - \frac{2}{y}t = -2y^3$$

$$\Rightarrow \quad ty^{-2} = -y^2 + c$$

$$\Rightarrow \quad \frac{x^2}{y^2} = -y^2 + c$$

also given that, this curve passes through (0, 2).

That is $\frac{0^2}{(2)^2} = -(2)^2 + c$ which yields $c = 4$.

$\therefore$ The desired equation is $\frac{x^2}{y^2} = -y^2 + 4$

Now $\{y(4)\}^2 \left\{4 - \{y(4)\}^2\right\} = M^2$

i. e., $M^2 = 4^2$

$$\Rightarrow \quad |M| = 4$$

3.Sol: Using lebnitz theorem, we get

$\sin(x).f(\sin(x)).\cos(x) = \cos(x)$

$$\Rightarrow \quad f(\sin(x)) = \frac{1}{\sin(x)}$$

Now $x = \frac{\pi}{6}$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) = 2 \Rightarrow \left\{f\left(\frac{1}{2}\right)\right\}^2 = 4$$

4.Sol: We have $m + m^2 = 2p$ and $m^3 = q$

$\left(m + m^2\right)^3 = p^3 8$

i.e., $m^3+m^6+3m^3\left(m+m^3\right)=8p^3$

$q+q^2+6pq=8p^3$

$\Rightarrow \quad \dfrac{q+q^2+6pq}{p^3}=8$

5.Sol: $a+b+c=8$

$ab+bc+ca=12$

$\Rightarrow \quad c=8-(a+b)0$

i.e., $ab+b\left[8-(a+b)\right]+a\left[8-(a+b)\right]=12$

$b^2+b(a-8)+a^2-8a+12=0$

also given $b\in R$, so $D\geq 0$

i.e., $(a-8)^2-4\left(a^2-8a+12\right)\geq 0$

$\Rightarrow \quad 3a^2-16a-16\leq 0$

Set of integral values of a is $\{0, 1, 2, 3, 4, 5, 6\}$.

$\therefore$ Number of integral values of a is 7.

6.Sol: Suppose normals at α,β,γ are concurrent at (h, k) and let s be the foot of the fourth normal from (h, k), then we have, $\sum\tan\left(\dfrac{\alpha}{2}\right)\tan\left(\dfrac{\beta}{2}\right)=0$ and

$\tan\left(\dfrac{\alpha}{2}\right)\tan\left(\dfrac{\beta}{2}\right)\tan\left(\dfrac{\delta}{2}\right)=-1.$

Eliminating $\tan\left(\dfrac{\delta}{2}\right)$ from above, we will get

$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$

7.Sol: Given $AQ=BP$

$\Rightarrow AB-AQ=AB-BP$

$=AP$

i.e., $BQ=AP$

Also given $2PM=PQ\Rightarrow PM=MQ$

i.e., $AP+PM=MQ+QB$

$\Rightarrow \quad AM=MB$

M is mid point of AB

$M=(2,3) \quad \Rightarrow a+b=2+3=5$

8.Sol:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\dfrac{\lambda k^4+2k^3+k^2+k+1}{n^5}}{\dfrac{3n^5+n^2+n+5k}{n^5}}=\int_0^1\frac{\lambda x^4}{3}dx=\frac{\lambda}{3}\left(\frac{x^5}{5}\right)_0^1=\frac{\lambda}{15}$$

Section-II:

1.Sol: $\int_0^x g(t)dt=2-k^2x^2=2-\int_0^x 2k^2t\,dt$

Let $f(x)=\int_0^x\left[2k^2t+g(t)\right]dt-2$

As, $2k^2t$ and $g(t)$ are continuous

$\Rightarrow\left[2k^2t+g(t)\right]$ is also continuous

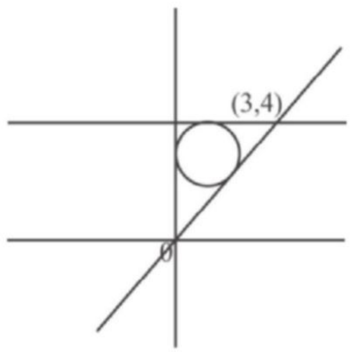
$f(0)=-2, \lim_{x\to\infty}f(x)\to\infty$

As $f(x)$ changes its sign

$\Rightarrow f(x)=0$ for same $x\in R$ and $\forall k\in R$

2.Sol: Given circle is $(x-1)^2+(y-3)^2=1$

Let tangent to the circle is $y-3=m(x-1)+\sqrt{1+m^2}$ and (3, 4) lies on it.



i.e., $1=2m+\sqrt{1+m^2}$

$\Rightarrow \quad (1-2m)^2=1+m^2$

$\Rightarrow \quad 4m^2-4m+1=1+m^2$

$\Rightarrow \quad m^2-4m=0$

i.e., $m=0,\dfrac{4}{3}$

$$\Rightarrow \quad \frac{y-4}{x-3} = m = 0, \text{ or } \frac{4}{3}$$

Smallest value is 0 and Largest value is $= \frac{4}{3}$

3.Sol: $\frac{1}{2}\operatorname{cosec}^2\left(x^2+y^2\right)d\left(x^2+y^2\right) = 2\left(\frac{y}{x}\right)^3 d\left(\frac{y}{x}\right)$

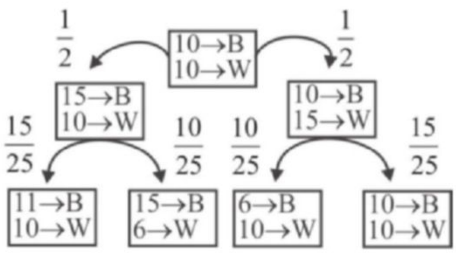
$$\Rightarrow -\frac{1}{2}\cot\left(x^2+y^2\right) - \frac{2(y/x)^4}{4} + c = 0$$

4.Sol: Roots of the equation

$y^3 - x_3y^2 - x_2y - x_1 = 0$ are 3, 5 and 7

i.e., $x_3 = 15, x_2 = -71, x_1 = 105$

5.Sol:



$$\frac{\frac{1}{2}\times\frac{15}{25}\times\frac{10}{21}+\frac{1}{2}\times\frac{10}{25}\times\frac{15}{21}}{\frac{1}{2}\times\frac{15}{25}\times\frac{10}{21}+\frac{1}{2}\times\frac{10}{25}\times\frac{6}{21}+\frac{1}{2}\times\frac{10}{25}\times\frac{15}{21}+\frac{1}{2}\times\frac{15}{25}\times\frac{11}{21}}$$

$$= \frac{2}{1+\frac{2}{5}+1+\frac{11}{10}} = \frac{20}{11+20+4} = \frac{4}{7}$$

6.Sol: Normal at $P(x_1, y_1)$

$$y - y_1 = \frac{-1}{m}(x - k_1)$$

Where $\quad y = 0, x = x_1 + my_1$

We have $\quad |x_1 + my_1| = 2|x_1|$

$$\Rightarrow x_1 + my_1 = \pm 2x_1$$

case-1: $x_1 + my_1 = 2x_1 \Rightarrow m = \frac{x_1}{y_1}$

$$\frac{dy}{dx} = \frac{x}{y}\left(\therefore m = \left(\frac{dy}{dx}\right)_{at(x_1y_1)}\right)$$

$$\Rightarrow x\,dx - y\,dy = 0; \frac{x^2}{2} - \frac{y^2}{2} = C - a \text{ hyperbola}$$

Case-II: $x_1 + my_1 = -2x_1; m = \frac{-3x_1}{y_1}$

$$y\frac{dy}{dx} + 3x = 0$$

$$\Rightarrow \quad \frac{y^2}{2} + \frac{3x^2}{2} = C \text{ (Ellipse)}$$

7.Sol: We know, $A.M \geq H.M$ for $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$,

we get, $$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \frac{3}{6}$$

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{2}$$

Now, $$\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2}{3}$$

$$\geq \left(\frac{a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}}{3}\right)^2 \geq \left(\frac{6+\frac{3}{2}}{3}\right)^2$$

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq \frac{75}{4}$$

8.Sol: $f(g(x)) = \sqrt{\frac{1}{x^2-x+1} - \left(x^2-x+1\right)}$

$$= \sqrt{\frac{1-\left(x^2-x+1\right)^2}{x^2-x+1}}$$

(a) Domain of $f(g(x)) : x - x^2 \geq 0 \Rightarrow x \in [0,1]$

(b) Range of $f(g(x)) : g(x) = x^2 - x + 1, x \in [0,1]$

$$\Rightarrow \quad g(x) = \left[\frac{3}{4}, 1\right]$$

Now, $\quad f(g(x)) = \sqrt{\frac{1}{g} - g}, g \in \left[\frac{3}{4}, 1\right]$

$\therefore f(g(x))$ is decreasing

$\Rightarrow$ Max.value of $f(g(x)) = \sqrt{\frac{7}{12}}$ at $x = \frac{3}{4}$

$\Rightarrow$ Min.value of $f(g(x)) = 0$ at $x = 1$

(c) Since, $g(x)$ is many one in [0, 1]

$\therefore f(g(x))$ is many one

(d) Hence $f(g(x))$ is bounded

Section III:
Paragraph - I

1,2.Sol: $f(x) = (x+a)(x^2 - 2ax + 1)$

$D < 0 \Rightarrow a \in (-1,1)$

and $f(x) = 0 \Rightarrow x = -a, a + \sqrt{a^2-1}, a - \sqrt{a^2-1}$

If $a > 1$, $\alpha < \beta < \gamma$

$\Rightarrow \alpha = -a$; $\beta = a - \sqrt{a^2-1}$ & $\gamma = a + \sqrt{a^2-1}$

$\Rightarrow \alpha < -1, \beta > 0, \gamma > 0$

Paragraph - II

3&4.Sol: Any two points on $y = x^2$ is

$P(\alpha, \alpha^2); Q(\beta, \beta^2)$

Equation of PQ, $y - \alpha^2 = (\alpha+\beta)(x-\alpha)$

$y = (\alpha+\beta)x - \alpha\beta$

Required area $\int_{\alpha}^{\beta}\left[(\alpha+\beta)x - \alpha\beta - x^2\right]dx$

$\Rightarrow \quad \beta - \alpha = 2$

Pair of tangents from origin are $y = 2x$ and $y = -2x$.

$\therefore$ Required Area $= \int_0^1\left[(x^2+1) - 2x\right]dx = \frac{2}{3}$

EVAAITS (JEE ADVANCED - 4) SOLUTIONS

ANSWER KEY

Section I:

1. c **2.** **3.** c **4.** a **5.** c
6. b

Section II:

1. b,c **2.** b,c **3.** a,c **4.** a,b **5.** b,c
6. b,c **7.** a,c **8.** a,c

Section III:

1. c **2.** a **3.** c **4.** d

HINTS & SOLUTIONS

Section I:

1. Sol: Put $i = j = k$, we get

$a_{ij} = 0$ and put $k = i \Rightarrow a_{ij} = -a_{ji}$

So, matrix is skew symmetric of odd order

2.Sol: Let $I = \int_2^4 x\,g(x)\,dx$

Using by parts, we get

$$I = x\,f^{-1}(x)\Big|_2^4 - \int_2^4 f^{-1}(x)\,dx$$

$$I = 4f^{-1}(4) - 2f^{-1}(2) - \int_2^4 f^{-1}(X)dx$$

Now, $\quad f^{-1}(f(x)) = x$

$\Rightarrow \quad f^{-1}(f(0)) = 0 \Rightarrow f^{-1}(2) = 0$

also $f^{-1}(f(1)) = 1 \Rightarrow f^{-1}(4) = 1$

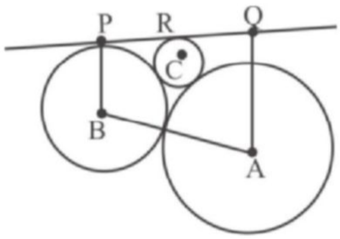
Now, $\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$

$\int_0^1 f(x)dx + \int_2^4 f^{-1}(x)dx = 1\cdot f(1) - 0\cdot f(0)$

$$\int_0^1 \left(x^3 + x + \sin \pi x + 2\right) dx + \int_2^4 f^{-1}(x)\, dx = 4 \text{ also,}$$

$$\int_2^4 f^{-1}(x)\, dx = 4 - \left(\frac{x^4}{4} + \frac{x^2}{2} - \frac{\cos \pi x}{\pi} + 2x\right)_0^1$$

3.Sol: We have, length of direct common tangent is $PQ^2 = 4r_1r_2$ and $QR^2 = 4r_1r_3$. From the diagram,



we have $PQ = PR + QR$

i.e., $$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

4.Sol: $S_n = 1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} ...... = \sum \frac{3^n - 1}{(3-1)n!}$

$$\frac{1}{2}\sum_{r=1}^{n} \frac{3^r - 1}{r!} = \frac{1}{2}\left[\sum \frac{3^r}{r!} - \frac{1}{r!}\right]$$

$$\lim_{n\to\infty}(S_n) = \frac{1}{2}\left[\left(e^3 - 1\right) - \left(e - 1\right)\right] = \frac{1}{2}\left[e^3 - e\right] \approx 8.59$$

$$[S_n] = 8$$

5.Sol: Put $x = \frac{1}{1-t}$ in l_2 and $x = 1 - \frac{1}{t}$ in l_3

We get $l = \int_{-20}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 \left(1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right) dx$

Let $u = \frac{x^3 - 3x + 1}{x(x-1)}$ then $l = \int \frac{du}{u^2}$

6.Sol: $P = \operatorname{cosec}\frac{\pi}{8} + \operatorname{cosec}\frac{2\pi}{8} + \operatorname{cosec}\frac{3\pi}{8}$

$$+ \operatorname{cosec}\left(2\pi - \frac{3\pi}{8}\right) + \operatorname{cosec}\left(2\pi - \frac{2\pi}{8}\right)$$

$$+ \operatorname{cosec}\left(2\pi - \frac{\pi}{8}\right) = 0$$

$$Q = 8\sin 10^\circ \sin 50^\circ \sin 70^\circ$$

$$= 8\sin 10^\circ \sin(60^\circ - 10^\circ)\sin(60^\circ + 10^\circ)$$

$$= 8\sin 30^\circ / 4 = 1$$

Section-II

1.Sol: Sum of roots =sum of diagonal elements and product of roots=value of the determinant.

2.Sol: $kx^2 + (2k-1)xy + y^2 + 2x - 2ky = 0$

$$a = k; b = 1; 2h = 2k - 1; g = 1; f = -k; c = 0$$

$$abc + 2fgh - bg^2 - ch^2 - af^2 = 0$$

i.e., $0 + (2k-1)\cdot(-k)\cdot 1 - 1\cdot 1 - 0 - k\cdot k^2 = 0$

$$\Rightarrow k^3 + 2k^2 - k + 1 = 0$$

Therefore, the product of roots is negative.

i.e., Atleast one negative root

Now $3k^3 + 1 = -6k^2 + 3k - 2$

$$D = 9 - 4 \times 6 \times 2 < 0$$

i.e., $3k^3 + 1 < 0$

For atleast one real value of k

3.Sol: $\overrightarrow{DA} = \vec{a}, \overrightarrow{DB} = \vec{b}, \overrightarrow{DC} = \vec{c}$

$$\overrightarrow{AD}.\overrightarrow{BC} = \left(-\vec{a}\right).\left(\vec{c} - \vec{b}\right)$$

$$= -\vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b}$$

$$= 0$$

Hence, $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{AB} \perp \overrightarrow{CD}$

Now, $\overrightarrow{PG} = -\frac{1}{6}\left(3\vec{a} - 2\vec{b} + \vec{c}\right) \& \overrightarrow{DC} = \frac{1}{2}\left(\vec{a} + \vec{b}\right)$

Let $\left|\vec{a}\right| = \left|\vec{b}\right| = \left|\vec{c}\right| = k$

$$\Rightarrow \left|\overrightarrow{PG}\right| = \frac{k}{2} \& \left|\overrightarrow{DQ}\right| = \frac{\sqrt{3}}{2}k$$

$$\Rightarrow \overrightarrow{PG}.\overrightarrow{DQ} = -\frac{1}{6}\left(3\vec{a} - 2\vec{b} + \vec{c}\right).\frac{1}{2}\left(\vec{a} + \vec{b}\right)$$

i.e., $\cos\theta = -\frac{5}{6\sqrt{3}} \Rightarrow \theta = \pi - \cos^{-1}\left(\frac{5}{6\sqrt{3}}\right)$

4.Sol: $$\frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{\sin x} + \frac{\frac{\sqrt{3}+1}{2\sqrt{2}}}{\cos x} = 2$$

$$\Rightarrow \sin\frac{\pi}{12}.\cos x+\cos\frac{\pi}{12}\sin x=\sin 2x$$

$$\Rightarrow \sin\left(x+\frac{\pi}{2}\right)=\sin 2x$$

$$n=\frac{\pi}{12} \text{ and } \frac{11\pi}{36}$$

5.Sol: $T_n=\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$ where $f(x)=x^4+x^3+x^2+2$,

$f(x)$ is an increasing function for $\forall x>0$.

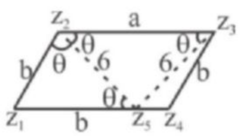
$$T_n=\frac{1}{n}\left[f(0)+f\left(\frac{1}{n}\right)+....+f\left(\frac{n-1}{n}\right)\right]$$

$$T_n<\int_0^1\left(x^4+x^3+x^2+2\right)dx=\frac{167}{60}$$

$$S_n=\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)>\frac{1}{n}\left[f\left(\frac{1}{n}\right)+....+f\left(\frac{r}{n}\right)\right]=\frac{167}{60}$$

6.Sol: $a=b+5$

and $\frac{b}{6}=\frac{6}{a}$



$$\Rightarrow ab=36$$

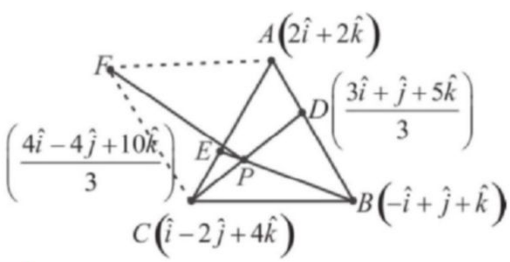
$$b^2+5b-36=0$$

$$\Rightarrow b=4$$

7.Sol: $CD:\vec{r}=\left(\hat{i}-2\hat{j}+4\hat{k}\right)+\frac{\lambda}{3}\left(7\hat{j}-7\hat{k}\right)$

$$BE:\vec{r}=\left(-\hat{i}+\hat{j}+\hat{k}\right)+\frac{\mu}{3}\left(7\hat{i}-7\hat{j}+7\hat{k}\right)$$

$$p\equiv\left(\hat{i}-\hat{j}+3\hat{k}\right)$$



Area of tetrahedron ABCF

$=\frac{1}{3}$ (Area of base triangle) $\times$ height $=\frac{7}{3}$ cubic units

$\overrightarrow{AB}\times\overrightarrow{AC}=7\hat{j}+7\hat{k},\left|\overrightarrow{PF}\right|=PF=\sqrt{2}$ units

$$PF=\sqrt{2}\left(\frac{7\hat{i}+7\hat{k}}{\sqrt{49+49}}\right)=\hat{j}+\hat{k}$$

= Position vector of F- position vector of P

$\therefore$ Position vector of F is $\hat{i}+4\hat{k}$

The equation of a vector AF is

$$\vec{r}=2\left(\hat{i}+\hat{k}\right)+\alpha\left(-\hat{i}+2\hat{k}\right)$$

8.Sol: If $\frac{A}{a},\frac{B}{b},\frac{C}{c}$ are in $H.P$, then

$$\frac{2b}{B}=\frac{a}{A}+\frac{c}{C}$$

$$\Rightarrow \quad 2bB=aC+cA$$

$$\Rightarrow \quad aB+cB=aC+cA$$

$$\Rightarrow \quad a[B-C]=c[A-B]$$

so, $r=\frac{c}{a}$

$\frac{A^2}{a},\frac{B^2}{b},\frac{C^2}{c}$ are in $H.P \Rightarrow r^2=\frac{C}{a}$

Section III:

Paragraph-I

1,2.Sol: $\sum_{i=1}^{n-1}\left(X_i-2i\sqrt{X_i-i^2}\right)=0$

$$\Rightarrow \sum_{i=1}^{n-1}\left(\sqrt{X_i-i^2}\right)^2-2i\sqrt{X_i-i^2}+i^2=0$$

$$\Rightarrow \sum_{i=1}^{n-1}\left(\sqrt{X_i-i^2-1}\right)^2=0$$

so, $X_i=2i^2$

Now, $X_1^2+.....+X_n^2=280$

$$\Rightarrow 2\left[1^2+2^2+....n^2\right]=280$$

$$n=7$$

$$Y_1 + Y_2 + Y_3 = 7$$

$$Y_1^1 + Y_2^1 + Y_3^1 = 4$$

$$^{4+3-1}C_3 = ^6C_3 = 20$$

Total triangles formed $= {}^{15}C_3 = \dfrac{15 \times 14 \times 13}{6}$

$\therefore$ No. of isosceles triangles formed $= 15 \times 7$

Probability $= \dfrac{15 \times 7}{15 \times 14 \times 13} \times 6 = \dfrac{3}{13}$

Paragraph -II

3,4.Sol: (I) $f(x) = \left\| x-6 \right| - \left| x-8 \right\| - \left| x^2 - 4 \right| + 3x - \left| x-7 \right|^3$

is continuous $\forall x \in R$ and not differentiable at $x = -2, 2, 6, 7$ and 8

(II) $f(x) = \left(x^2 - 9\right)\left|x^2 + 11x + 24\right| + \sin\left|x-7\right|$

$+ \cos\left|x-4\right| + (x-1)^{\frac{3}{5}} \sin(x-1)$

is continuous $\forall x \in R$ and not differentiable at $x = -8, 7$

$$f(x) = \begin{cases} (x+1)^{3/5} - \dfrac{3\pi}{2} & : x < -1 \\ \left(x - \dfrac{1}{2}\right)\cos^{-1}\left(4x^3 - 3x\right) & : -1 \le x \le 1 \\ (x-1)^{5/3} & : 1 < x < 2 \end{cases}$$

is discontinuous at $x = -1$, and 1, and not differentiable at $x = -1, -\dfrac{1}{2}, 1$

(IV) $f(x) = \{\sin x\}\{\cos x\} + \{\sin^3 \pi\{x\}([x]), x \in [-1, 2\pi]$

Let $g(x) = \left(\sin \pi \{x\}\right)([x])\left(\sin^2 \pi\{x\}\right)$

$g'(1^+) = g'(1^-)$ so differentiable at $x = 1$ and for $\{\sin x\} \cdot \{\cos x\}$ doubtful points for non differentiability are $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

$\{\sin x\} \cdot \{\cos x\}$ is discontinuous at $x = 0, \dfrac{\pi}{2}, 2\pi$

So, it is not differentiable at $x = 2n\pi, 2n\pi + \dfrac{\pi}{2}$

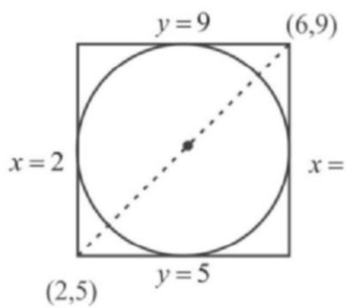
EVAAITS (BIT-SAT - 1) SOLUTIONS

ANSWER KEY

1. d	2. a	3. c	4. b	5. c
6. d	7. d	8. b	9. b	10. c
11. d	12. c	13. c	14 .c	15. a
16. b	17. b	18. c	19. d	20. b
21. a	22. d	23. d	24. c	25. c
26. c	27. d	28. c	29. b	30. b
31. d	32. b	33. b	34. c	35. c
36. b	37. c	38. a	39. b	40. c
41. c	42. a	43.a	44. a	45. a

HINTS & SOLUTIONS

1.Sol: We can see from the graph that center of the inscribed circle and mid-point of the diagonals are equal.



i.e., centre is (4,7)

2.Sol: Let A $(ae, 0)$ and $B(-ae, 0)$ be two given points and (h, k) be the coordinates of the moving point P.

Now, $PA+PB=2a$

$$\Rightarrow \sqrt{(h-ae)^2+k^2}+\sqrt{(h+ae)^2+k^2}=2a \quad (1)$$

But, we know that

$$\left[(h-ae)^2+k^2\right]-\left[(h+ae)^2+k^2\right]=-4aeh \quad (2)$$

Dividing (1) by (2), we get

$$\sqrt{\left[(h-ae)^2+k^2\right]}-\sqrt{\left[(h+ae)^2+k^2\right]}=-2eh \quad (3)$$

Adding (1) and (3),

$$2\sqrt{\left[(h-ae)^2+k^2\right]}=2(a-eh)$$

Squaring upon both sides, we get

$$\Rightarrow (h-ae)^2+k^2=(a-eh)^2 \Rightarrow \frac{h^2}{a^2}+\frac{k^2}{a^2(1-e^2)}=1$$

Hence locus of P is $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

3.Sol: Given expression $(1+x+x^2+....)^2$

$$=\left[(1-x)^{-1}\right]^2=(1-x)^{-2}$$

$$=(1+2x+3x^2+4x^3+....+(n+1)x^n+nx^{n-1}+....)$$

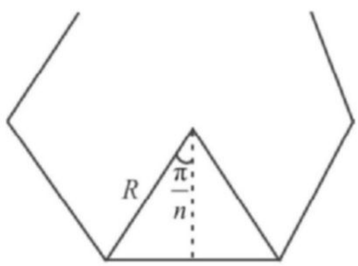
therefore coefficient of x^n is $(n+1)$.

4.Sol: Total number of elements in a set is 4

In cartesian product, we have 16 ordered pairs and we know number of reflexive relation is 2^{n^2-n}

i.e., $2^{16-4}=2^{12}$

5.Sol: From the diagram, we have $\frac{r}{R}=\cos\left(\frac{\pi}{n}\right)$



i.e., If $\frac{r}{R}=\frac{1}{2}$, then $n=3$, likewise

$\frac{r}{R}=\frac{1}{\sqrt{2}} \Rightarrow n=4$, $\frac{r}{R}=\frac{\sqrt{3}}{2} \Rightarrow n=6$ and for $\frac{r}{R}=\frac{2}{3}$

$\Rightarrow$ any integer values for n.

6.Sol: Given $\sin x+\cos x=\frac{1}{5}$

upon squaring on both sides,

we get, $\Rightarrow \sin^2 x+\cos^2 x+2\sin x\cos x=\frac{1}{25}$

i.e., $\sin 2x=\frac{-24}{25} \Rightarrow \cos 2x=\frac{-7}{25} \Rightarrow \tan 2x=\frac{24}{7}$

7.Sol: Given here, $\sin x+\cos x=\min_{a\in IR}\{1, a^2-4a+6\}$

$$a^2-4a+6=a^2-4a+4+2=(a-2)^2+2\geq 2$$

$$\therefore \sin x+\cos x=1$$

Now, $\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$

$$\Rightarrow \sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \Rightarrow x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$$

$$\therefore x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$$

8.Sol: Let S be the sample space for a random experiment and let E be the event then complement of event E is

$$n(S)-n(E)=n(\overline{E})$$

$$\therefore \quad n(E)+n(\overline{E})=n(S)$$

i.e., probability that first plane misses

$=1-$ probability the first plane hits

$=1-0\cdot 3=0\cdot 7$

probability that seconds plane hits is 0.2

The desired probability is $0\cdot 7\times 0\cdot 2$

$=0\cdot 14$

9.Sol: Minor of $-4=\begin{vmatrix}-2 & 3\\ 8 & 9\end{vmatrix}=-42$,

Minor of $9=\begin{vmatrix}-1 & -2\\ -4 & -5\end{vmatrix}=-3$ and cofactor of

$-4=(-1)^{2+1}(-42)=42$,

$\therefore$ cofactor of 9 is $(-1)^{3+3}(-3)=-3$

10.Sol: $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$$\frac{dy}{dx}=\frac{a\sin t}{a(1-\cos t)}=\cot\left(\frac{t}{2}\right)$$

11.Sol: No. of ways in which 4 boys can be seated is $3!=6$

Also given that there are 4 different coloured chairs.

$\therefore$ Total number of ways $=6\times 4=24$

12.Sol: The equation of an ellipse is $\frac{x^2}{9}+\frac{y^2}{16}=1$

Here, $a^2=9$ and $b^2=16$

$\therefore$ The equation of the auxiliary circle is

$$x^2+y^2=9.$$

13.Sol: $T=S_1$ is the equation of desired chord, hence

$$xx_1+yy_1-a^2=x_1^2+y_1^2-a^2 \Rightarrow xx_1+yy_1=x_1^2+y_1^2$$

14.Sol: We have $x^2+y^2=9$ (1)

and $x+y=3$ (2)

From equation (1) and (2), we make a homogeneous equation

i.e., $x^2+y^2=(x+y)^2$

$\Rightarrow x^2+y^2=x^2+y^2+2xy$

$\Rightarrow 2xy=0$

i.e., $xy=0$

15.Sol: Given $(1+x)^n=C_0+C_1x+C_2x^2+...+C_nx^n$

Multiply 'x' through out the equation, we get

$$x(1+x)^n=C_0x+C_1x^2+C_2x^3+...+C_nx^{n+1}$$

Differentiating with respect to x,

we get

$$(1+x)^n+nx(1+x)^{n-1}=C_0+2C_1x+3C_2x^2+...+n+1C_nx^n$$

Put $x=1$

$$\Rightarrow 2^n+n2^{n-1}=C_0+2C_1+3C_2+...+(n+1)C_n$$

i.e., $C_0+2C_1+3C_2+...+(n+1)C_n=2^{n-1}(n+2)$

16.Sol: $f(x)=\log(x+\sqrt{x^2+1})$

and $f(-x)=-\log(x+\sqrt{x^2+1})=-f(x)$

$\therefore$ $f(x)$ is odd function

17.Sol: $\cot B+\cot C=\frac{\sin(B+C)}{\sin B\cdot\sin C}$

$$=\frac{\sin(180-A)}{\sin B.\sin A}$$

$$=\frac{\sin A}{\sin B\sin C}$$

similarly, $\cot C+\cot A=\frac{\sin B}{\sin C\cdot\sin A}$

and $\cot A+\cot B=\frac{\sin C}{\sin(A)\sin B}$

Therefore, $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot B)$

$$=\frac{\sin A\cdot\sin B\cdot\sin C}{\sin^2 A\cdot\sin^2 B\cdot\sin^2 C}$$

$$=\operatorname{cosec}A\ \operatorname{cosec}B\ \operatorname{cosec}C$$

18.Sol: Let $a=3x+4y, b=4x+3y$ and $c=5x+5y$.

Clearly, C is the largest side and thus the largest angle C is given by

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{-2xy}{2(12x^2+25xy+12y^2)}<0$$

$\therefore$ C is obtuse angle.

19.Sol: The required probability

$$={}^8C_6\left(\frac{1}{2}\right)^6\cdot\left(\frac{1}{2}\right)^2+{}^8C_7\left(\frac{1}{2}\right)^7\cdot\left(\frac{1}{2}\right)+{}^8C_8\left(\frac{1}{2}\right)^8=\frac{37}{256}$$

20.Sol: $s(t)=at^3+bt+5$ in metre in t (sec)

$$\therefore \frac{ds(t)}{dt}=3at^2+b$$

$$\therefore \frac{d^2\{s(t)\}}{dt^2}=6at$$

$$\therefore 6at\big|_{t=4}=48$$

$$\Rightarrow a=2$$

21.Sol: Let edge of the cube be x cm.

Volume of the cube be x^3cm^3

Given, $\frac{dx}{dt}=10cm/\sec$

Now, $V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$

$$\Rightarrow \frac{dV}{dt} = 3(5)^2(10)cm^3/\sec = 750cm^3/\sec$$

22.Sol: $\frac{^nP_r}{^nC_r} = 24 \Rightarrow r! = 24 \Rightarrow r = 4$

$\therefore {}^nC_4 = 35 \Rightarrow n = 7.$

23.Sol: Asymptotes are given by $9x^2 - 25y^2 = 0$.

Therefore, equation of the hyperbola has equation of the form $9x^2 - 25y^2 = k$

Now, vertices are $(\pm 5, 0)$

So, putting $y = 0, x = \sqrt{\frac{k}{9}}$

Thus, $k/9 = 25 \Rightarrow k = 225$

So, required equation is $9x^2 - 25y^2 = 225$

24.Sol: $x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$

$(1+\lambda)x^2 + (1+\lambda)y^2 - 2x - 4y + (1-\lambda) = 0$

$$\Rightarrow \quad x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0 \qquad (1)$$

$\therefore$ Centre is $\left(\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right)$

and radius is

$$\sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{1+\lambda}$$

Since it touches the line $x + 2y = 0$

i.e., $$\left|\frac{\frac{1}{1+\lambda} + 2\frac{2}{1+\lambda}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{1+\lambda}$$

$$\Rightarrow \quad \lambda = \pm 1.$$

$\lambda = -1$ cannot be possible in case of circle. So

$\lambda = 1 \qquad (1)$

Thus, from(1) $x^2 + y^2 - x - 2y = 0$ is the required equation of the circle.

25.Sol: $f(0^+) = \lim_{x\to 0^+}\left[\frac{\sin x}{x} + \cos x\right] = 1 + 1 = 2$

and $f(0^-) = \lim_{x\to 0^-}\left[\frac{\sin x}{x} + \cos x\right] = 1 + 1 = 2$

and $f(0) = 2$

Hence $f(x)$ is continuous at $x = 0$

26.Sol: $$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$C_1 \to C_1 - 4C_2 - 3C_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

[Replace C_1 and C_2 and then Replace C_2 by C_3]
Hence rank of matrix is 2.

27.Sol: $x^2 = -4y \Rightarrow 2x = -4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-x}{2}$

i.e., $\left(\frac{dy}{dx}\right)_{(-4,-4)} = 2$

We know that equation of tangent is,

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$$

$$\Rightarrow y + 4 = 2(x+4) \Rightarrow 2x - y + 4 = 0$$

28.Sol: Given curve is $y^4 = ax^3 \Rightarrow 4y^3\frac{dy}{dx} = 3ax^2$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,a)} = \frac{3a^3}{4a^3} = \frac{3}{4}$$

$\therefore$ Equation of normal at point (a, a) is

$$y - a = -\frac{4}{3}(x - a) \Rightarrow 4x + 3y = 7a.$$

29.Sol: As sum of coefficients is zero, hence one root is l and other root is $\frac{l-m}{m-n}$

Also given, it has equal roots

$$\frac{l-m}{m-n}=l \Rightarrow 2m=n+l$$

30.Sol: $$\frac{1+\frac{2^2}{2!}+\frac{2^4}{3!}+\frac{2^6}{4!}+....\infty}{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+...\infty}=$$

$$=\frac{\frac{1}{2^2}\left\{\frac{2^2}{1!}+\frac{(2^2)^2}{2!}+\frac{(2^2)^3}{3!}+....\right\}}{\frac{1}{2^2}\left\{2+2+\frac{2^2}{2!}+\frac{2^3}{3!}+....\right\}}=\frac{e^{(2^2)}-1}{1+e^2}=e^2-1$$

31.Sol: We know that the expression $ax^2+bx+c>0$ for all x, if $a>0$ and $b^2<4ac$

$\therefore (a^2-1)x^2+2(a-1)x+2>0, \forall x$

if $a^2-1>o$ and $4(a-1)^2-8(a^2-1)<0$

i.e., $a^2-1>0$ and $(a-1)(a+3)>0$

$\Rightarrow a^2>1$ and $a<-3$ or $a>1 \Rightarrow a<-3$ or $a>1$

32.Sol: $$\int\frac{dx}{1+3\sin^2 x}=\int\frac{dx}{\sin^2 x+\cos^2 x+3\sin^2 x}$$

$$=\int\frac{\sec^2 x dx}{4\tan^2 x+1}=\frac{1}{4}$$

Put $t=\tan x \Rightarrow dt=\sec^2 x dx$, then it reduce to

$$\frac{1}{4}\int\frac{dt}{t^2+\left(\frac{1}{2}\right)^2}=\frac{1}{4}2\tan^{-1}(2t)+c$$

$$=\frac{1}{2}\tan^{-1}(2\tan x)+c$$

33.Sol: ${}^nC_2=66 \Rightarrow n(n-1)=132 \Rightarrow n=12.$

34.Sol: Here, $g_1=\frac{k}{2}, f_1=2, c_1=2,$

$$g_2=-1, f_2=\frac{-3}{4}, c_2=\frac{k}{2}$$

we have, Condition for orthogonal intersection is

$$\Rightarrow 2(g_1g_2+f_1f_2)=c_1+c_2$$

$$\Rightarrow 2\left[\frac{-k}{2}+\left(\frac{-3}{2}\right)\right]=2+\frac{k}{2}$$

$$\Rightarrow -k-3=2+\frac{k}{2} \Rightarrow k=\frac{-10}{3}$$

35.Sol: L.H.L. $\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{\tan h}{h}=1$

R.H.L. $\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{\tan h}{h}=1$

$\therefore$ L.H.L.=R.H.L.$=f(0)=1$

Continuous at x=0

Now L.H.D. $\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\tan h-h}{-h^2}$

$$=\lim_{h\to 0}\frac{\sec^2 h-1}{-2h}$$

$$=\lim_{h\to 0}\frac{2\sec^2 h\tan h}{-2}=0$$

and R.H.D. $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\frac{\tan(0+h)}{(0+h)}-1}{h}$

$$=\lim_{h\to 0}\frac{2\sec^2 h.\tan h}{2}=0$$

$\therefore$ L.H.D.=R.H.D. $\Rightarrow$ Differentiable at $x=0$

$\therefore f(x)$ is both continuous and differentiable at $x=0$.

36.Sol: Let $I=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{\left(\frac{k}{n}\right)}{1+\left(\frac{k}{n}\right)^2}$

$$I=\int_0^1\frac{x}{1+x^2}dx=\frac{1}{2}\left[\log(1+x^2)\right]_0^1=\frac{1}{2}[\log 2]$$

37.Sol: $\frac{10}{9},\frac{1}{3}\sqrt{\frac{20}{3}},\frac{2}{3}....$

$\therefore r = \frac{1}{3}\sqrt{\frac{20}{3} \cdot \frac{9}{10}} = \frac{\sqrt{60}}{10} = \sqrt{\frac{3}{5}}$

$\therefore ar^4 = \frac{10}{9} \times \left(\frac{3}{5}\right)^2 = \frac{2}{5}$.

38.Sol: $\tan^{-1}\left\{\frac{1}{\sqrt{\cos\alpha}} - \tan^{-1}\left[\sqrt{\cos\alpha}\right]\right\} = x$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1 + \frac{\sqrt{\cos\alpha}}{\sqrt{\cos\alpha}}}\right\} = x$$

i.e., $\tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

$\therefore \quad \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2\left(\frac{\alpha}{2}\right)$

39.Sol: $\frac{\lim\limits_{x\to 0} \frac{\sin mx}{mx}.mx}{\lim\limits_{x\to 0} \frac{\tan nx}{nx}.nx} \Rightarrow \frac{m}{n}$

40.Sol: $6 = \frac{3+4+x+7+10}{5} \Rightarrow 30 = 24 + x \Rightarrow x = 6$

41.Sol: $x_1, x_2, x_3 \ldots \infty$

$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right) \ldots\ldots \infty$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\right) + \sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\right)$

$= \cos\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) + i\sin\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right) = \cos\pi + i\sin\pi = -1$

42.Sol: Given series is

$3 + 4\frac{1}{2} + 6\frac{3}{4} + \ldots = 3 + \frac{9}{2} + \frac{27}{4} + \ldots$

$= 3 + \frac{3^2}{2} + \frac{3^3}{4} + \frac{3^4}{8} + \frac{3^5}{16} + \ldots$ (in G.P)

Here $a = 3$, $r = \frac{3}{2}$, then sum of the five terms is

$$S_5 = \frac{a\left(r^n - 1\right)}{r-1} = \frac{3\left[\left(\frac{3}{2}\right)^5 - 1\right]}{\frac{3}{2} - 1}$$

$= 6\frac{[243-32]}{32} = \frac{633}{16} = 39\frac{9}{16}$

43.Sol: $\cos x dy = y(\sin x - y)dx$

$\Rightarrow \frac{dy}{dx} = \frac{y\sin x - y^2}{\cos x} \Rightarrow \frac{dy}{dx} = y\tan x - y^2 \sec x$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \qquad (1)$

Put $\frac{1}{y} = t$ in equation (1)

$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

From equation (1) and (2), we get

$\frac{dt}{dx} + t.\tan x = \sec x$

$\therefore I \cdot F = e^{\int \tan x\, dx} = e^{\log|\sec x|} = \sec x$

$\therefore$ Solution of differential equation is

$t.\sec x = \int \sec x.\sec x.dx + c$

$\Rightarrow \frac{1}{y}\sec x = \tan x + c \Rightarrow \sec x = y(\tan x + c)$

44.Sol: Given $H = \frac{2pq}{p+q}$

$\Rightarrow \frac{H}{P} + \frac{H}{q} = \frac{2q}{p+q} + \frac{2p}{p+q} = \frac{2(p+q)}{p+q} = 2$

45.Sol: $\sum(2n-1)^3 = \sum(8n^3 - 3\cdot 4n^2 + 3.2n - 1)$

$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$

$= 2n^4 - n^2 = n^2(2n^2 - 1)$

EVA AITS (BITSAT - 2) SOLUTIONS

ANSWER KEY

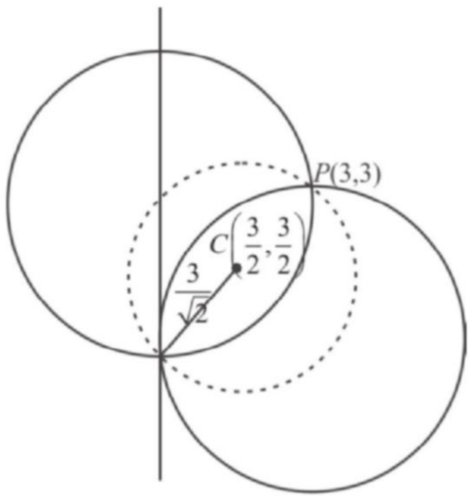
1. c	2. c	3. a	4. b	5. a
6. b	7. c	8. c	9. b	10. a
11. b	12. c	13. b	14. b	15. a
16. c	17. b	18. b	19. d	20. b
21. b	22. b	23. d	24. d	25. a
26. c	27. c	28. a	29. b	30. d
31. b	32. c	33. b	34. d	35. c
36. c	37. d	38. a	39. a	40. d
41. d	42. d	43. b	44. a	45. b

HINTS & SOLUTIONS

1.Sol: Given $x^2+y^2-6x=0$ and $x^2+y^2-6y=0$

Clearly from the graph, we can see that these two curves intersects at either origin or at (3,3)

$\therefore$ Equation of the desired circle is



$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\left(0-\frac{3}{2}\right)^2+\left(0+\frac{3}{2}\right)^2$$

$$\Rightarrow \quad x^2+y^2-3x-3y=0$$

2.Sol: We know, tan of half of the angle between two tangents is the ratio of radius to the length of a tangent.

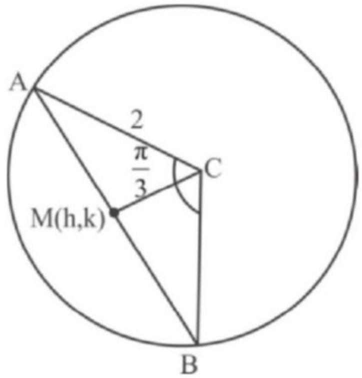
i.e., $$\tan\frac{\theta}{2}=\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}$$

$$\Rightarrow \frac{\theta}{2}=\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$$

$$\therefore \quad \theta=2\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$$

3.Sol: The centre of given circle is (1,1) and its radius is $\sqrt{2}$.

From the figure, if $M(h,k)$ be the middle point of chord AB subtending an angle $\frac{2\pi}{3}$ at C, then

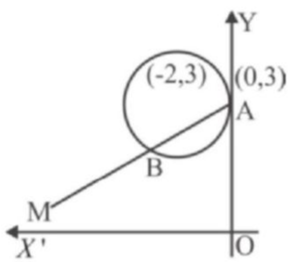


$$\frac{CM}{AC}=\cos\frac{\pi}{3}=\frac{1}{2}\Rightarrow 4CM^2=AC^2$$

i.e., $4[(h-1)^2+(k-1)^2]=4\Rightarrow h^2+k^2-2h-2k+2=1$

Hence the locus is $x^2+y^2-2x-2y+1=0$.

4.Sol:



$$\left(\frac{h}{2}\right)^2+4\left(\frac{h}{2}\right)+\left(\frac{k+3}{2}-3\right)^2=0$$

$$\Rightarrow \frac{h^2}{4}+\frac{8h}{4}+\frac{(k-3)^2}{4}=0$$

or $x^2+y^2+8x-6y+9=0$, which is a circle.

5.Sol: Using $P+\lambda Q=0$, the required line is $12x-y-31=0$ and its distance from both the points is $\frac{31}{\sqrt{145}}$.

6.Sol: Bisector of the angle between positive directions of the axes is $y=x$. Since it is one of the lines of the given pair $ax^2+2hxy+by^2=0$, we have

$x^2(a+2h+b)=0$ or $a+b=-2h$

7.Sol: For continuity at $x=0$, we must have

$$f(0)=\lim_{x\to 0} f(x)=\lim_{x\to 0}(x+1)^{\cot x}$$

$$=\lim_{x\to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{\lim_{x\to 0}\left(\frac{x}{\tan x}\right)}=e$$

8.Sol: Conceptual.

9.Sol: Given α,β are the roots of the equation

$x^2-2x\cos\phi+1=0$

i.e., $\therefore\ x=\frac{2\cos\phi\pm\sqrt{4\cos^2\phi-4}}{2}$

$\Rightarrow \alpha=\cos\phi+i\sin\phi=e^{i\phi},\beta=\cos\phi-i\sin\phi=e^{-i\phi}$

$\therefore\ \alpha^n=(e^{i\phi})^n=e^{in\phi};\beta^n=e^{-in\phi}$

$x^2-(\alpha^n+\beta^n)x+\alpha^n.\beta^n=0$

$\Rightarrow x^2-(e^{in\phi)}+e^{-in\phi})x+1=0$

$\Rightarrow x^2-2\cos n\phi.x+1=0$

10.Sol: Given equation is $x^2-2ax+a^2+a-3=0$

If roots are real, then $D\geq 0$

i.e., $\Rightarrow 4a^2-4(a^2+a-3)\geq 0$

$\Rightarrow a-3\leq 0\Rightarrow a\leq 3$

As roots are less tahn 3, hence $f(3)>0$

$9-6a+a^2+a-3>0\Rightarrow a^2-5a+6>0$

$\Rightarrow (a-2)(a-3)>0\Rightarrow$ either $a<2$ or $a>3$

Hence $a<2$ satisfy all.

11.Sol: $(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+....+C_nx^n$

$(1-x)^n=C_0-C_1x+C_2x^2-C_3x^3+....+(-1)^nC_nx^n$

$[(1+x)^n-(1-x^n)]=2[C_1x+C_3x^3+C_5x^5+..]$

$1/2[(1+x)^n-(1-x^n)]=C_1x+C_3x^3+C_5x^5+....$

Put $x=2,\ 2C_1+2^3.C_3+2^5.C_5+....=\frac{3^n-(-1)^n}{2}$

12.Sol: Putting $x=1$ in $(1+x-3x^2)^{2163}$

We get sum of the coefficients as

$(1+1-3)^{2163}=(-1)^{2163}=-1$

13.Sol: $A\cap X=B\cap X=\phi$

$\therefore\ A$ and X,B and X are disjoint sets

Also, $A\cup X=B\cup X\Rightarrow A=B$

14.Sol: $(gof)(x)=|\sin x|$ and $f(x)=\sin^2 x$

$\Rightarrow g(\sin^2 x)=|\sin x|;\ \therefore\ g(x)=\sqrt{x}$.

15.Sol: $\tan\frac{P}{2}+\tan\frac{Q}{2}=-\frac{b}{a}$

$\tan\frac{P}{2}\tan\frac{Q}{2}=\frac{c}{a}\Rightarrow\tan\left(\frac{P+Q}{2}\right)=\frac{-b/a}{1-c/a}$

as $P+Q=\frac{\pi}{2}\Rightarrow 1=\frac{-b}{a-c}\Rightarrow c=a+b$

16.Sol: $\cos A=\frac{3}{5}\Rightarrow\sin A=-\frac{4}{5}$

$\cos B=\frac{4}{5}\Rightarrow\sin B=-\frac{3}{5}$

$\therefore\ 2\left(-\frac{4}{5}\right)+4\left(-\frac{3}{5}\right)=-\frac{20}{5}=-4$

17.Sol: $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$

$$f_4-f_6=\frac{1}{4}(\sin^4 x+\cos^4 x)-\frac{1}{6}(\sin^6 x+\cos^6 x)$$

$$=\frac{1}{4}(1-2\sin^2 x\cos^2 x)-\frac{1}{6}(1-3\sin^2 x\cos^2 x)$$

$$\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$$

18.Sol: $\cot\theta=\sin 2\theta,(\theta\neq n\pi)\Rightarrow 2\sin^2\theta\cos\theta=\cos\theta$

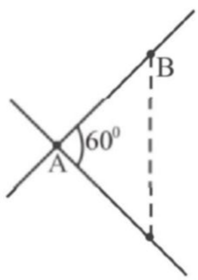
$$\Rightarrow \cos\theta=0 \text{ or } \sin^2\theta=\frac{1}{2}=\sin^2\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \theta=(2n+1)\frac{\pi}{2} \text{ or } \theta=n\pi\pm\frac{\pi}{4}$$

$\therefore \theta=90^\circ$ and 45°

19.Sol: Let the two roads intersect at A. If the bus and the car are at B and C on the two roads respectively, then $c=AB=2km, b=AC=3km$. The distance between the two vehicles

$= BC = a\,km$



Now $\cos A=\cos 60^0=\dfrac{b^2+c^2-a^2}{2bc}$

$$\Rightarrow \frac{1}{2}=\frac{3^2+2^2-a^2}{2\cdot 3\cdot 2}\Rightarrow a=\sqrt{7}\ km$$

20.Sol: There are 366 days in a leap year, in which 52 weeks and two days, there are 7 combinations of 2 days, among them 2 are favourable for 53 fridays and 2 for saturdays.

i.e., P(53 Fridays) $=\dfrac{2}{7}$; P(53 Saturdays) $=\dfrac{2}{7}$

P(53 Fridays and 53 Saturdays) $=\dfrac{1}{7}$

$\therefore$ P(53 Fridays or Saturdays) = P(53 Fridays) + P(53 Saturdays) - P(53 Fridays and Saturdays)

$$=\frac{2}{7}+\frac{2}{7}-\frac{1}{7}=\frac{3}{7}$$

21.Sol: Required probability = Probability that either the number is 7 or the number is 8.

i.e., Required Probability $=P_7+P_8$

Now $P_7=\dfrac{1}{2}\cdot\dfrac{1}{11}+\dfrac{1}{2}\cdot\dfrac{6}{36}=\dfrac{1}{2}\left(\dfrac{1}{11}+\dfrac{1}{6}\right)$

$$P_8=\frac{1}{2}\cdot\frac{1}{11}+\frac{1}{2}\cdot\frac{5}{36}=\frac{1}{2}\left(\frac{1}{11}+\frac{5}{36}\right)$$

$$\therefore\quad P=\frac{1}{2}\left(\frac{2}{11}+\frac{11}{36}\right)=0.244$$

22.Sol: Given, that A,B,C are angles of triangle.

We have $A+B+C=\pi$,

$\Rightarrow A+B=\pi-c$

$\Rightarrow \cos(A+B)=\cos(\pi-C)=-\cos C$

i.e., $\cos A\cos B-\sin A\sin B=-\cos C$

$\Rightarrow \cos A\cos B+\cos C=\sin A\sin B$

and $\sin(A+B)=\sin(\pi-C)=\sin C$

Expanding the given determinant, we get

$$\Delta=-(1-\cos^2 A)+\cos C(\cos C+\cos A\cos B)$$
$$+\cos B(\cos B+\cos A\cos C)$$
$$=-\sin^2 A+\sin A(\sin B\cos C+\cos B\sin C)$$
$$=-\sin^2 A+\sin A\sin(B+C)=-\sin^2 A+\sin^2 A=0$$

23.Sol: The coefficient determinant

$$D=\begin{vmatrix}2 & -1 & -1\\ 1 & -2 & 1\\ 1 & 1 & \lambda\end{vmatrix}=-3\lambda-6$$

For no solution, the necessary condition is $D=0$

i.e., $-3\lambda-6=0\Rightarrow\lambda=-2$

It can be seen that for $\lambda=-2$, there is a solution for the given system of equations.

24.Sol: $P^T=2P+I\Rightarrow(P^T)^T=(2P+1)^T$

$\Rightarrow P=2P^T+I\Rightarrow P=2(2P+I)+I$

$\Rightarrow 3P=-3I\Rightarrow P=-I\Rightarrow PX=-IX=-X$

25.Sol: $y\sqrt{x^2+1}=\log\left\{\sqrt{x^2+1}-x\right\}$

$$\Rightarrow\frac{dy}{dx}\sqrt{x^2+1}+y\cdot\frac{2x}{2\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}-x}\left\{\frac{1}{2}\frac{2x}{\sqrt{x^2+1}}-1\right\}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx}+xy=\sqrt{x^2+1}\cdot\frac{-1}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx}+xy+1=0$$

26.Sol: $f'(x)=2x-6=0 \Rightarrow x=3$

27.Sol: $a+bv^2=x^2 \Rightarrow 0+b\left(2v\cdot\frac{dv}{dt}\right)=2x\cdot\frac{dx}{dt}$

$$\Rightarrow v\cdot b\frac{dv}{dt}=x\cdot\frac{dx}{dt}\Rightarrow\frac{dv}{dt}=\frac{x}{b}$$

28.Sol: Given curve: $y=a(e^{x/a}+e^{-x/a})$

Now, for tangent to be parallel to $x-$axis, slope must equal to zero.

i.e., $\frac{dy}{dx}=0$

Now, $\frac{dy}{dx}=\frac{d}{dx}\left[ae^{x/a}+ae^{-x/a}\right]=a\cdot e^{x/a}\frac{1}{a}+ae^{-x/a}\left(\frac{-1}{a}\right)$

$$\Rightarrow\frac{dy}{dx}=e^{x/a}-e^{-x/a}=0$$

i.e., $e^{x/a}=e^{-x/a}\Rightarrow e^{2(x/a)}=1=e^{\circ}$

$\Rightarrow \quad x=0$

29.Sol: $f(x)=\int_{-10}^{x}(t^4-4)e^{-4t}dt\Rightarrow f'(x)=(x^4-4)e^{-4x}$

Now $f'(x)=0\Rightarrow x=\pm\sqrt{2}$

Now $f''(x)=-4(x^4-4)e^{-4x}+4x^3e^{-4x}$

At $x=\sqrt{2}$ and $x=-\sqrt{2}$ the given function has extreme value.

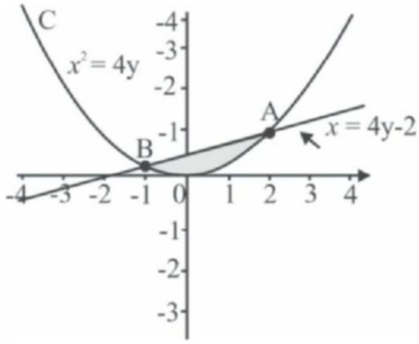
30.Sol: Put $x=\sin\theta\Rightarrow dx=\cos\theta\, d\theta$, therefore

$$\int\sin^{-1}(3x-4x^3)dx=\int\sin^{-1}(\sin 3\theta)\cos\theta\, d\theta$$

$$=\int 3\theta\cos\theta\, d\theta$$

$$=3\{\theta\sin\theta+\cos\theta\}+c=3\left\{x\sin^{-1}x+\sqrt{1-x^2}\right\}+c$$

31.Sol: Solving the equations $x^2=4y$ and $x=4y-2$ simultaneously. The points of intersection of the parabola and the line are $A(2,1)$ and $B\left(-1,\frac{1}{4}\right)$.



$\therefore$ The required area = shaded area

i.e., $\int_{-1}^{2}\frac{x+2}{4}dx-\int_{-1}^{2}\frac{x^2}{4}dx$

$$=\frac{1}{4}\left[\frac{x^2}{2}+2x\right]_{-1}^{2}-\frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}=\frac{9}{8}\,sq.unit$$

32.Sol: Required number of ways

$={}^4C_1\times{}^8C_5+{}^4C_2\times{}^8C_4+{}^4C_3\times{}^8C_3+{}^4C_4\times{}^8C_2$

$=4\times 56+6\times 70+4\times 56+1\times 28=896$

33.Sol: From the question, we have 2^{n-1} students gave wrong answer to atleast one question, like wise 2^{n-2} students gave wrong answer to atleast 2 questions and so on. Hence there is one student who answered all n questions wrong. i.e., 2°.

Now, total number of incorrect answers is

$$2^{n-1}+2^{n-2}+\ldots+2^{\circ}=2^{n-1}$$

If we set that to 2047, we get

$$2^n=2048 \text{ or } n=11.$$

34.Sol: Rewriting the expressions as

$={}^nC_r+2.{}^nC_{r-1}+{}^nC_{r-2}$

$=\left({}^nC_r+{}^nC_{r-1}\right)+\left({}^nC_{r-1}+{}^nC_{r-2}\right)$

$={}^{n+1}C_r+{}^{n+1}C_{r-1}={}^{n+2}C_r$

35.Sol: Locus of point of intersection of perpendicular tangent is directrix of the parabola.

So, $x=-1$.

36.Sol: Semi latus rectum is harmonic mean between segments of focal chords of a parabola.

$\therefore\quad b=\frac{2ac}{a+c}\Rightarrow a,b,c$ are in H.P.

37.Sol: We know $S_n = \frac{n}{2}\{2P+(n-1)Q\}$, hence $d = Q$.

38.Sol: $\frac{x^{n+1}+y^{n+1}}{x^n+y^n} = \sqrt{xy} \Rightarrow x^{n+1}+y^{n+1} = \sqrt{xy}(x^n+y^n)$

$$\Rightarrow x^{n+\frac{1}{2}}\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right) = y^{n+\frac{1}{2}}\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right) \Rightarrow \left(\frac{x}{y}\right)^{n+\frac{1}{2}} = 1$$

$$\Rightarrow n = -\frac{1}{2}$$

39.Sol: Let S_n be the sum of the given series to n terms, then

$S_n = 1+2x+3x^2+4x^3+....+nx^{n-1}$ (1)

$x\,S_n = x+2x^2+3x^2+....+nx^n$ (2)

Subtracting (1) from (2), we get

$(1-x)S_n = 1+x+x^2+x^3+....$ to n terms $-n\,x^n$

$$= \left(\frac{(1-x^n)}{(1-x)}\right) - nx^n$$

$$= \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$$

40.Sol: $1-\cos\alpha = \frac{1}{2-\sqrt{2}} = 1+\frac{1}{\sqrt{2}}$

$$= \cos\alpha = -\frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{3\pi}{4}$$

41.Sol: We have $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$$\Rightarrow \quad -\frac{\pi}{2} \le 2\sin^{-1} 2a \le \frac{\pi}{2}$$

i.e., $\frac{-\pi}{4} \le \sin^{-1} 2a \le \frac{\pi}{4}$

$$\Rightarrow \quad \frac{-1}{\sqrt{2}} \le 2a \le \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad |a| \le \frac{1}{2\sqrt{2}}$$

42.Sol: $\lim_{x\to 0}\frac{2}{x}\log(1+x) = \lim_{x\to 0} 2\log(1+x)^{\frac{1}{x}}$

$$= \lim_{x\to 0} 2\log_e e = 2$$

43.Sol: $\frac{dy}{dx} = y+2x$, Put $y+2x = z \Rightarrow \frac{dy}{dx}+2 = \frac{dz}{dx}$

$$\therefore \quad \frac{dz}{dx}-2 = z \Rightarrow \frac{dz}{dx} = z+2$$

$$\Rightarrow \int\frac{dz}{z+2} = \int dx \Rightarrow \log(z+2) = x+c$$

$$\Rightarrow \log(y+2x+2) = x+c \Rightarrow y+2x+2 = e^{x+c}$$

For initial value, $2 = e^c \Rightarrow c = \log_e 2$

$\therefore$ Solution is $y+2x+2 = 2e^x$

$$\Rightarrow y = 2(e^x - x - 1)$$

44.Sol: $\bar{x}$ for population $A = \frac{101+102+...+200}{100}$

$$= 150.5$$

$\bar{x}$ for population $B = \frac{151+152+...+250}{100}$

$$= 200.5$$

$$V_A = \frac{(101-150.5)^2+(102-150.5)^2+...+(200-150.5)^2}{100}$$

$$= \frac{(49.5)^2+(48.5)^2+...+(0.5)^2+(0.5)^2+(1.5)^2+...+(49.5)^2}{100}$$

$$V_B = \frac{(151-200.5)^2+...+(250-200.5)^2}{100}$$

$$= \frac{(49.5)^2+....+(0.5)^2+(0.5)^2+...+(49.5)^2}{100}$$

$$\Rightarrow \quad \frac{V_A}{V_B} = 1$$

45.Sol: Given $|8+z|+|z-8| = 16$

Clearly locus of z is an ellipse.